COLLATZ MEETS FIBONACCI

MICHAEL ALBERT, BJARKI GUDMUNDSSON AND HENNING ULFARSSON

A well-known conjecture due to Lothar Collatz from 1937 states that when the function

$$f(x) = \begin{cases} x/2 & \text{if } x \equiv 0 \pmod 2 \\ 3x+1 & \text{if } x \equiv 1 \pmod 2 \end{cases}$$

is iterated on an initial positive integer x we eventually reach the cycle $(1, 4, 2)$. Although many have tried to prove the conjecture, or find a counterexample to it, it remains open to this day. For this work we assume that the conjecture is true. The collection [1] contains many articles related to the conjecture and its generalizations, and an annotated bibliography. One of the intriguing features of this function is that, starting from some initial value x, the sequence of iterates: $x, f(x), f(f(x)), \ldots,$ behaves at first irregularly before its eventual apparent inevitable decline from some power of 2 down to the three element cycle. Here we consider the initial interesting phase of these iterations when seen from a distance, i.e. only with regard to their relative values, and not the actual numbers produced.

For example, starting the iteration at 12 provides the sequence:

$$\underbrace{12,\ 6,\ 3,\ 10,\ 5}_{\text{trace}},\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \ldots\ .$$

The sequence of numbers up until the first power of two is the interesting phase of the iteration, which we will call its *trace*. The elements of a trace are all distinct, and viewing it from a distance we might replace each element of a trace by its rank (i.e. the i^{th} smallest number of the trace is replaced by i). The resulting sequence is a permutation, and we shall call permutations produced in this manner *Collatz permutations*. We denote the Collatz permutation obtained from initial value x by $C(x)$. So we have: $C(12) = 5\,3\,1\,4\,2$. It seems natural to ask: *Among the permutations of length n, how many are Collatz permutations?*

Considering only those $x \le 10^8$ for which the length of $C(x)$ is at most 7 produces Table 1. As the reader will have noticed the values in the table are the Fibonacci

Length	Collatz permutations
1	1
2	1
3	2
4	3
5	5
6	8
7	13

TABLE 1. Number of Collatz permutations of length less than 8 (experimentally).

numbers. In what follows we will explain this phenomenon and show that it persists through length 14. Beyond that point, excess permutations appear – and we will explain how and why this occurs as well.

1. Types of traces

The appearance of the Fibonacci numbers in the enumeration of Collatz permutations is easy to explain. The steps that occur in a trace can be: *up steps* ($x \mapsto 3x+1$ when x is odd) denoted by u; and *down steps* ($x \mapsto x/2$ when x is even) denoted by d. Two up steps can never occur consecutively since $3x+1$ is even when x is odd. The step types in a trace can be recovered from the resulting Collatz permutation according to the pattern of rises and descents. We call the resulting sequence of u's and d's the *type* of the trace. As well as not containing consecutive u's, the last symbol in such a sequence must be a d (since there is a "hidden" u occurring next to take us to a power of 2). As is well known, the number of such sequences of length n is given by F_n, the n^{th} Fibonacci number (with $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$).

So, in order to show that there are at least F_n Collatz permutations of length n it will be enough to show that any sequence of u's and d's satisfying the necessary conditions above actually occurs as the type of some trace. To that end let a *witness* for a type, σ, be an $A = 2^a$ such that there is a trace ending at $(A-1)/3$ with type σ.

The following proposition shows that every potential type has a witness and thereby proves that there are at least F_n Collatz permutations of length n.

Proposition 1. *If a type σ contains k u's then there is a single congruence of the form $A = c \pmod{3^{k+1}}$ which must be satisfied in order that a trace of type σ ends with witness A. Consequently, there is a least witness $A = 2^a$ with $a \leq 2 \cdot 3^k$, and a general witness is of the form 2^{a+jd} where j is a nonnegative integer and $d = 2 \cdot 3^k$.*

Table 2 shows that there are exactly F_n Collatz permutations of length n for $n = 1, 2, \ldots, 14$ but for greater n there are more.

length	#perms	length	#perms	excess
1	1	15	611	1
2	1	16	989	2
3	2	17	1600	3
4	3	18	2587	3
5	5	19	4185	4
6	8	20	6771	6
7	13	21	10953	7
8	21	22	17720	9
9	34	23	28669	12
10	55	24	46383	15
11	89	25	75044	19
12	144	26	121417	24
13	233	27	196448	30
14	377	29	317850	39

Table 2. Number of Collatz permutations of length less than 30

The first type that is associated with more than one Collatz permutation is the type $\sigma = uddudududdudd d$ which has the integrality condition $2^a = 16 \bmod 729$. The smallest solution to this equation is $a = 4$ corresponding to the trace $9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5$ and the first permutation in Figure 1. However, the next solution to the integrality condition is $a = 490$, giving an initial number with 440 digits. This initial number produces a different permutation than the smaller initial number, and is shown in the second line in Figure 1.

3	12	7	2	10	5	13	8	15	11	6	14	9	4	1
4	12	7	2	10	5	13	8	15	11	6	14	9	3	1

FIGURE 1. Two different permutations associated with the type *uddudududdudd d*

In the next section we will explain why this type gives us two different permutations and when this should be expected.

2. Excess permutations

How can one type correspond to different permutations? We show that a type σ with n symbols gives rise to $n+1$ linear functions. For example the type $\sigma = dududd$ gives the lines in Figure 2. We show that a witness for the type corresponds to wherever we find a vertical line $t = A$ where $A = 2^a$ and all the intersection points of $t = A$ with these lines are at integer heights. We can then read off the relative order of the lines at this point according to the order they are crossed as we move up the line $t = A$ from the t-axis.

Consequently we see that if we have two potential witnesses such that there is no intersection point between two lines in the corresponding family lying between them, that they will determine the same permutation. On the other hand, if there

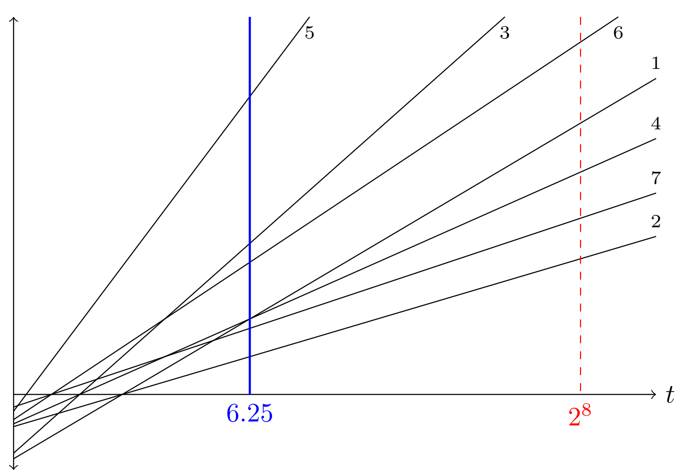

FIGURE 2. Linear functions determining permutations associated with the type *dududd*. The labels on each line correspond to the point that the corresponding element would occur in a trace and so the permutation associated with the intersections on the line $x = 2^8$ is 4 1 6 3 7 5 2.

were a witness between every pair of intersections of the lines for a type, and if these intersections all occurred at distinct abscissae, we might have up to $\binom{n}{2}$ witnesses for any given type producing distinct permutations. However, we can rule out such a wealth of witnesses quite easily as we can show that the second potential witness always lies to the right of the rightmost intersection point among the lines. That is:

Proposition 2. *For any type σ there are at most two distinct permutations $C(x)$ arising from x of type σ.*

Propositions 1 and 2 show that the number of Collatz permutations of length n lies between F_n and $2F_n$.

To get an exact enumeration of the Collatz permutations one would need to understand which types are associated with two permutations. We call these *excess creating types* (*ET's*). Given an ET we can always create a new ET by prepending a d. This is because the extra d does not alter the integrality condition and can only increase the maximum intersection point. This at least shows that the number of ET's is non-decreasing.

3. Future work

As with so many aspects of the whole Collatz disease, a few answers just seem to lead to more questions.

- How exactly does the number e_n of ET's of length n behave? The data above suggests that it might be something like "half Fibonacci rate" i.e. $e_n \sim e_{n-2} + e_{n-4}$.
- In our current dataset we always get $c = 16$ in the integrality conditions for ET's. Is this just bias in the data that we currently have? Is it a necessary condition? Or sufficient?
- We have an extrinsic way of creating the Collatz permutations: run the Collatz process and see what comes out. Is there an intrinsic way to recognize these permutations, beyond the obvious condition that they cannot contain consecutive rises?
- There are several other maps similar to the Collatz map and there is also the modified Collatz sequence (where u is replaced with ud), as well as the Syracuse function where long down-steps are collapsed into one down-step. How do these analyses transfer to those contexts?

For the reader who is eager to start exploring Collatz permutations, we have a small code library supporting some basic features for working with their types. The code is written in the Sage open-source mathematics software system, but should run in Python with minor modifications. The code can be found at `https://github.com/SuprDewd/CollatzPermutations`.

4. Acknowledgment.

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References

[1] Jeffrey C. Lagarias, editor. *The ultimate challenge: the* $3x+1$ *problem*. American Mathematical Society, Providence, RI, 2010.