Permutations arising from the Collatz-conjecture and automatic discovery of patterns

MIT Combinatorics Seminar

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First four sections are joint work with Michael Albert (Otago) and Bjarki Gudmundsson (Reykjavik)

The Collatz process

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x	$f(x)$	step
12	6	d

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x	$f(x)$	step
12	6	d
6	3	d

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x	$f(x)$	step
12	6	d
6	3	d
3	10	u

The Collatz process

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x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d

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$$f(x) = \begin{cases} x/2 & \text{if } x \equiv 0 \pmod 2 \\ 3x+1 & \text{if } x \equiv 1 \pmod 2 \end{cases}$$

x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5	16	u

The Collatz process

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x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5	16	u
16	8	d

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12	6	d
6	3	d
3	10	u
10	5	d
5	16	u
16	8	d
8	4	d

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x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5	16	u
16	8	d
8	4	d
4	2	d

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x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5	16	u
16	8	d
8	4	d
4	2	d
2	1	d

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12	6	d
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1		

The Collatz conjecture

The conjecture

The Collatz process ends in 1 for any starting number

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The Collatz process ends in 1 for any starting number

- First stated by Lothar Collatz in 1937
- Many have tried, but the conjecture remains open
- Starting numbers up to $5 \times 2^{60} \approx 5.764 \times 10^{18}$ have been verified

The initial permutation

x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5	16	u
16	8	d
8	4	d
4	2	d
2	1	d
1		

Discard powers of 2.

The initial permutation

x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5		

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The initial permutation

x	$f(x)$	step
12	6	d
6	3	d
3	10	u
10	5	d
5		

Discard powers of 2. The remaining numbers are distinct so we can flatten them to a permutation

The initial permutation

$$12 \quad 6 \quad 3 \quad 10 \quad 5$$

Put 1 instead of the smallest number (3), 2 instead of the next smallest (5), etc.

The initial permutation

$$\begin{matrix} 12 & 6 & 3 & 10 & 5 \\ & & 1 & & \end{matrix}$$

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$$\begin{array}{ccccc} 12 & 6 & 3 & 10 & 5 \\ & & 1 & & 2 \end{array}$$

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12	6	3	10	5
	3	1		2

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$$\begin{array}{ccccc} 12 & 6 & 3 & 10 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{array}$$

Put 1 instead of the smallest number (3), 2 instead of the next smallest (5), etc.

How many?

By running computer tests we get the following data, which we can think of as a lower bound on the correct enumeration

length	#perms
1	1

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1	1
2	1
3	2
4	3
5	5
6	8

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length	#perms
1	1
2	1
3	2
4	3
5	5
6	8
7	13

How many?

By running computer tests we get the following data, which we can think of as a lower bound on the correct enumeration

length	#perms
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21

How many?

By running computer tests we get the following data, which we can think of as a lower bound on the correct enumeration

length	#perms
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34

How many?

By running computer tests we get the following data, which we can think of as a lower bound on the correct enumeration

length	#perms
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55

How many?

By running computer tests we get the following data, which we can think of as a lower bound on the correct enumeration

length	#perms
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89

Possible operation sequences

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Now we need to show that each operation sequence is witnessed by some starting number

Looking from the tail

Let $X = 2^x$ be the number we hit in the tail. Let $D = d^{-1}$, $U = u^{-1}$ and consider the operation sequence $duddudud$:

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$$UDU(X) = (2X - 5)/9$$

Looking from the tail

Let $X = 2^x$ be the number we hit in the tail. Let $D = d^{-1}$, $U = u^{-1}$ and consider the operation sequence $duddudud$:

$$U(X) = (X - 1)/3$$
$$DU(X) = (2X - 2)/3$$
$$UDU(X) = (2X - 5)/9$$
$$DUDU(X) = (4X - 10)/9$$
$$UDUDU(X) = (4X - 19)/27$$
$$DUDUDU(X) = (8X - 38)/27$$
$$DDUDUDU(X) = (16X - 76)/27$$
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X needs to satisfy $32X = 206 \bmod 81$, or $X = 52 \bmod 81$

The modular requirement

- We had $X = 2^x$ so we get $2^x = 52 \bmod 81$

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Therefore every possible operation sequence is witnessed by some starting number so each one gives rise to one permutation, ...

Longer permutations

length n	#perms	$\text{fib}(n)$
10	55	55
11	89	89

Longer permutations

length n	#perms	$\text{fib}(n)$
10	55	55
11	89	89
12	144	144

Longer permutations

length n	#perms	$\mathrm{fib}(n)$
10	55	55
11	89	89
12	144	144
13	233	233

Longer permutations

length n	#perms	$\text{fib}(n)$
10	55	55
11	89	89
12	144	144
13	233	233
14	377	377

Longer permutations

length n	#perms	$\text{fib}(n)$
10	55	55
11	89	89
12	144	144
13	233	233
14	377	377
15	611	610

Longer permutations

length n	#perms	$\text{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1

Longer permutations

length n	#perms	$\text{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2

Longer permutations

length n	#perms	$\text{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3

Longer permutations

length n	#perms	$\mathrm{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3
18	2587	2584	3

Longer permutations

length n	#perms	$\mathrm{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3
18	2587	2584	3
19	4185	4181	4

Longer permutations

length n	#perms	$\mathrm{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3
18	2587	2584	3
19	4185	4181	4
20	6771	6765	6

Longer permutations

length n	#perms	$\mathrm{fib}(n)$	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3
18	2587	2584	3
19	4185	4181	4
20	6771	6765	6
21	10953	10946	7

Longer permutations

length n	#perms	fib(n)	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3
18	2587	2584	3
19	4185	4181	4
20	6771	6765	6
21	10953	10946	7
22	17720	17711	9

Longer permutations

length n	#perms	fib(n)	excess
10	55	55	
11	89	89	
12	144	144	
13	233	233	
14	377	377	
15	611	610	1
16	989	987	2
17	1600	1597	3
18	2587	2584	3
19	4185	4181	4
20	6771	6765	6
21	10953	10946	7
22	17720	17711	9
23	28669	28657	12

The Lines

Consider again

$$U(X) = (X-1)/3$$
$$DU(X) = (2X-2)/3$$
$$UDU(X) = (2X-5)/9$$
$$DUDU(X) = (4X-10)/9$$
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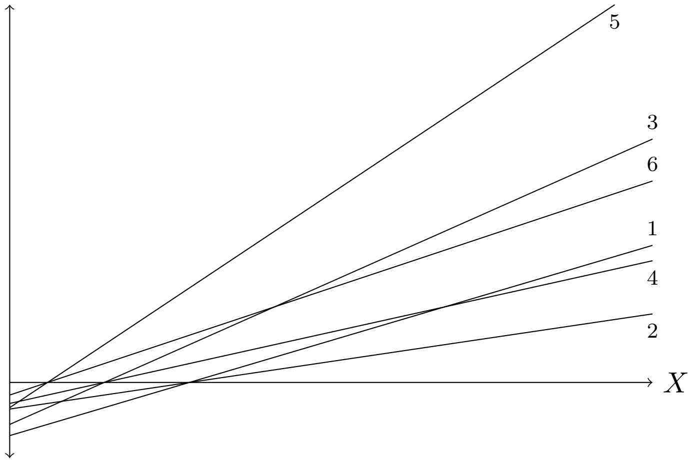
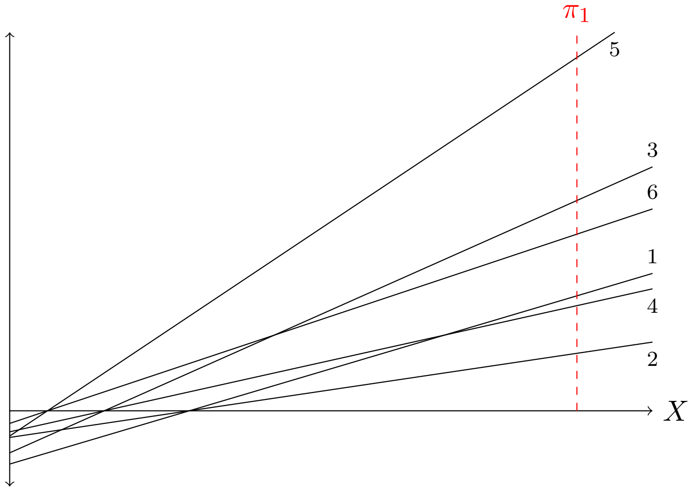
These are equations for lines

The Lines

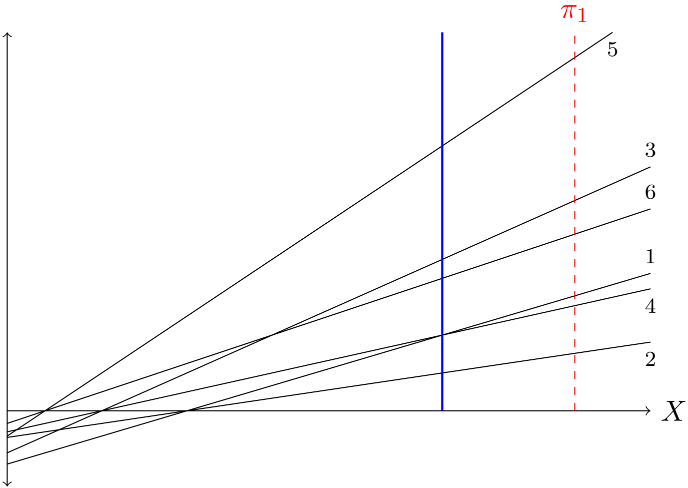


The Lines



The order of the lines determines the permutation.

The Lines



The order of the lines determines the permutation. Intersection points change that order.

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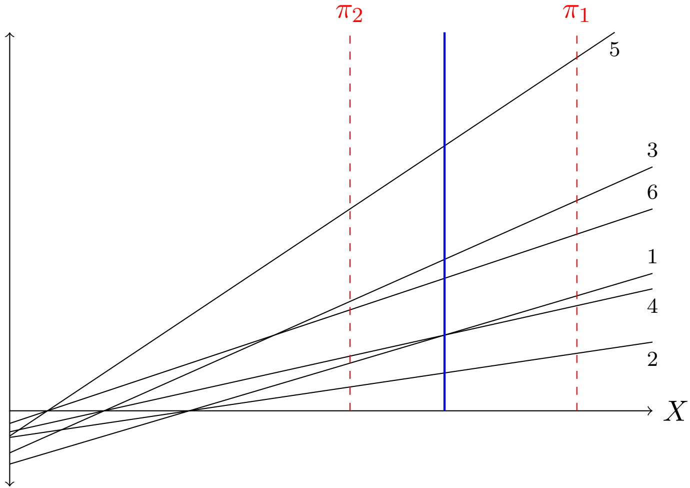

The order of the lines determines the permutation. Intersection points change that order.

Solutions within the largest intersection

Solutions to $2^x = c \mod 3^k$, for a fixed operation sequence, are what give us permutations.

- There is always a solutions outside the largest intersection point
- Between adjacent intersection points there could be a solution, giving us a different permutation
- This first occurs for an operation sequence of length 14

The first excess permutation

- The operation sequence $uddudududdu ddd$ gives us the modular requirement

$$2^x = 16 \bmod 729$$

The first two solutions are $x = 4$ and $x = 490$

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  - The solution $x = 4$ corresponds to a process ending in 2^4

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The first two solutions are $x = 4$ and $x = 490$

- The solution $x = 4$ corresponds to a process ending in 2^4
- The solution $x = 490$ corresponds to a process ending in 2^{490}

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$$2^x = 16 \bmod 729$$

  The first two solutions are $x = 4$ and $x = 490$
  - The solution $x = 4$ corresponds to a process ending in 2^4
  - The solution $x = 490$ corresponds to a process ending in 2^{490}
- The largest intersection point is ≈ 44.05 so these solutions give us different permutations

1	4	9	14	6	11	15	8	13	5	10	2	7	12	3
1	3	9	14	6	11	15	8	13	5	10	2	7	12	4

How many excess permutations?

- Current data points to the excess being close to

$$\sqrt{\operatorname{fib}(n-11)} \text{ for } n \geq 15$$

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- When an excess permutation appears it becomes a root of a tree of longer excess permutations
- We can get a very rough upper bound of $n^2\mathrm{fib}(n)$, since n lines can have at most n^2 intersection points
- But we can do better

An upper bound

It is not difficult to show that for an operation sequence of length $n-1$ with k u-steps the largest intersection point is bounded by $X = 2 \cdot 3^{k-1}$.

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The modular requirement has modulus $m = 3^k$ so the distance between two consecutive solutions is $\phi(m) = 2 \cdot 3^{k-1}$, so there can be at most one solution in the interval $[0, x]$, giving us $\mathrm{fib}(n)$ as an upper bound on the excess

Structure

- We do know a little about the structure, e.g., how a permutation of length n can be used to create permutations of length $n+1$ and $n+2$

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- We do know a little about the structure, e.g., how a permutation of length n can be used to create permutations of length $n+1$ and $n+2$
- A popular way to describe the structure of a class of permutations is to describe the patterns that the class avoids

Characterizing by what's not there

This is similar to other fields of mathematics

- planar graphs are the graphs that avoid K_5 and $K_{3,3}$

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- planar graphs are the graphs that avoid K_5 and $K_{3,3}$
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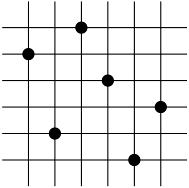
For a given class of permutations we want to find the patterns being avoided

Drawing permutations

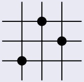
We can draw the graph of a permutation by placing dots on a grid

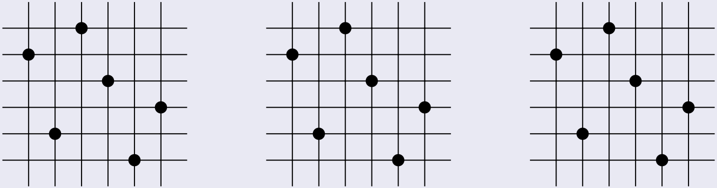
$526413 =$ 

Classical patterns

Patterns are permutations inside other permutations ...

Example

The pattern $132 =$  occurs in the permutation 526413



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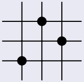
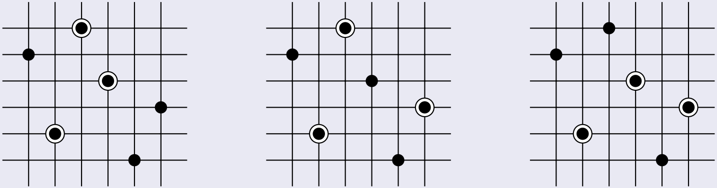
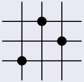
The pattern $132 =$  occurs in the permutation 526413

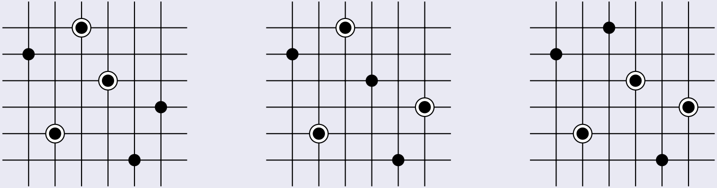


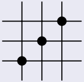
Classical patterns

Patterns are permutations inside other permutations . . .

Example

The pattern $132 =$  occurs in the permutation 526413



The same permutation avoids the pattern $123 =$ 

Some avoidance theorems

We use $\mathrm{Av}(P)$ to denote the perms avoiding the pattern(s) in P.
The class of

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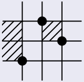
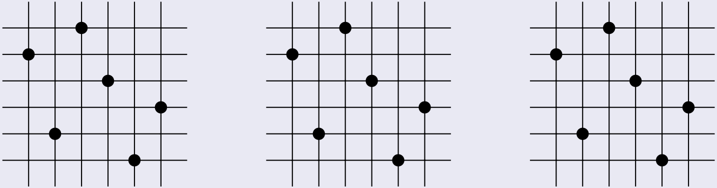
But this vocabulary is not powerful enough to describe West-2-stack-sortable perms, factorial Schubert varieties, alternating subgroup of perms, simsun perms, and others

Mesh patterns patterns

Mesh patterns, introduced by Claesson and Brändén, can restrict regions in the permutation

Example

The mesh pattern  occurs in the permutation 526413



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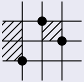
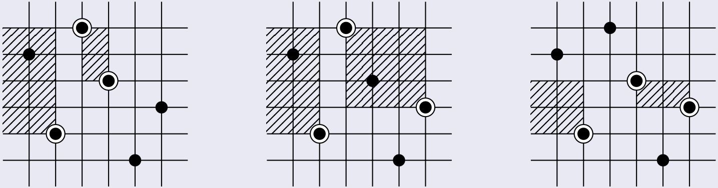
The mesh pattern  occurs in the permutation 526413



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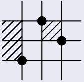
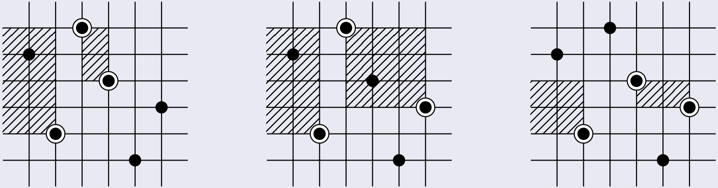
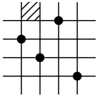
The mesh pattern  occurs in the permutation 526413



Only the last occurrence is valid.

Some more avoidance theorems

The class of

- West-2-stack-sortable perms $= \mathrm{Av}\Big(2341,$  $\Big)$ (West)

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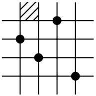
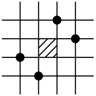
- West-2-stack-sortable perms $= \mathrm{Av}\left(2341, \right.$  $\left.\right)$ (West)
- factorial Schubert varieties $= \mathrm{Av}\left(1324, \right.$  $\left.\right)$

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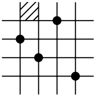
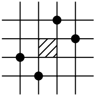
- West-2-stack-sortable perms $= \mathrm{Av}\left(2341,\right.$  $\left.\right)$ (West)
- factorial Schubert varieties $= \mathrm{Av}\left(1324,\right.$  $\left.\right)$ (Bousquet-Mélou+Butler)
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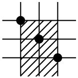
- simsun perms $= \mathrm{Av}\left(\right.$  $\left.\right)$ (Claesson+Brändén & U)

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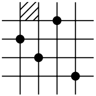
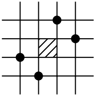
- West-2-stack-sortable perms $= \mathrm{Av}\left(2341,\right.$  $\left.\right)$ (West)
- factorial Schubert varieties $= \mathrm{Av}\left(1324,\right.$  $\left.\right)$ (Bousquet-Mélou+Butler)
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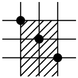
- simsun perms $= \mathrm{Av}\left(\right.$  $\left.\right)$ (Claesson+Brändén & U)

It is easy to see that any class of permutations can be described as avoiding a (possibly infinite) list of mesh patterns

Kid-in-the-candy-store problem

There are so many interesting classes of permutations! Which ones have nice descriptions with mesh patterns?

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More precicely: Construct an algorithm that inputs a (finite piece of a) class of perms and outputs a conjectural description in terms of mesh patterns

A classical algorithm

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$$12, 21,$$

Output: $\mathrm{Av}\left(\quad \right)$

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We need something more powerful if the class is avoiding mesh patterns

The idea behind BiSC

Bi = Billey, S = Steingrímsson, C = Claesson

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- Step 1: Look at the permutations in the class, to figure out the allowed patterns

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- Step 3: Clean up redundancies

Demo

You can download BiSC from my website:
http://staff.ru.is/henningu/programs/bisc/bisc.html
Here we will test the following classes

- West-2-stack-sortable perms
- dihedral subgroup
- alternating subgroup
- simsun perms

A closer look at step 1

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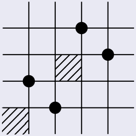

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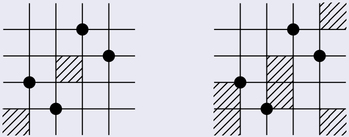

and then later a permutation containg the second pattern, we forget the first one, since it is now redundant

A closer look at step 2

- Step 2: Generate the forbidden patterns from the allowed patterns

A closer look at step 2

- Step 2: Generate the minimal forbidden patterns from the allowed patterns

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Example

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A closer look at step 2

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Example

For example, if we have the two allowed shadings of the classical pattern 12:

then we generate

A closer look at step 3

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There are several possibilities

- Try every subset of patterns in the output (bad if large output)

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- Apply the shading lemma (bad if large output)

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- Use the perms not in the class to see which patterns must be used (bad if large output)
- Apply the shading lemma (bad if large output)

Natural question

When does a set of mesh patterns M make a mesh pattern m redundant, i.e., $\mathrm{Av}\,(M) = \mathrm{Av}\,(M \cup \{m\})$?

Further work on BiSC

- Find a clever way to perform the clean-up step

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- Can BiSC prove theorems, instead of just stating conjectures?

Thanks!
Please ask questions!