Enumeration of Permutation Classes by Inflation of Independent Sets of Graphs

Émile Nadeau
(based on joint work with Christian Bean and Henning Ulfarsson)

Reykjavik University

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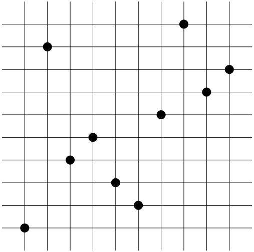
October 13, 2019

Definition (Permutation)

A *permutation* is considered to be an arrangement of the numbers $1, 2, \ldots, n$ for some positive n.



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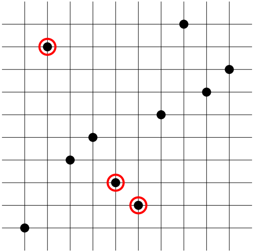
Definition (Pattern)

A permutation, or *pattern*, π is said to be contained in an other permutation σ if σ contains a subsequence order isomorphic to π.



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Definition

A *permutation class* is the set of permutations that avoid a given set of patterns. A permutation class is denoted $\mathrm{Av}(\sigma_1, \ldots, \sigma_n)$

Example

$$\mathrm{Av}(123) = \{\varepsilon, 1, 12, 21, \cancel{123}, 213, 231, 312, 321, \ldots\}$$

Generating function

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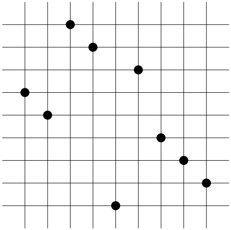
$$\frac{1}{1-x} = \sum_{n\geq 0} \frac{1}{n!} \left(\frac{1}{1-x}\right)^{(n)}\Big|_{x=0} x^n = \sum_{n\geq 0} \frac{1}{n!} \frac{n!}{(1-x)^{n+1}}\Big|_{x=0} x^n = \sum_{n\geq 0} x^n$$

Staircase encoding

For any permutation π we can extract the left-to-right minima and place them on the diagonal of a square grid.

Example

$\pi = \underline{6}\underline{5}98\underline{1}7432$



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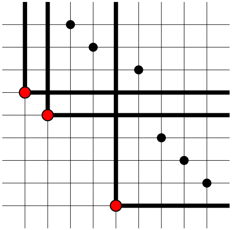

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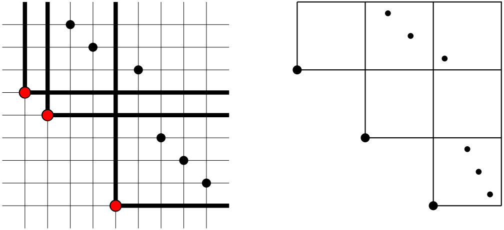

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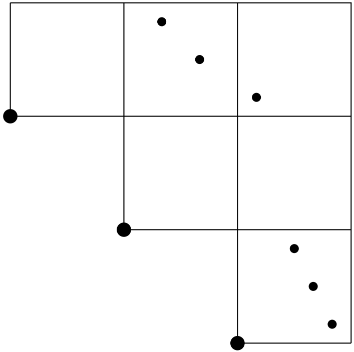

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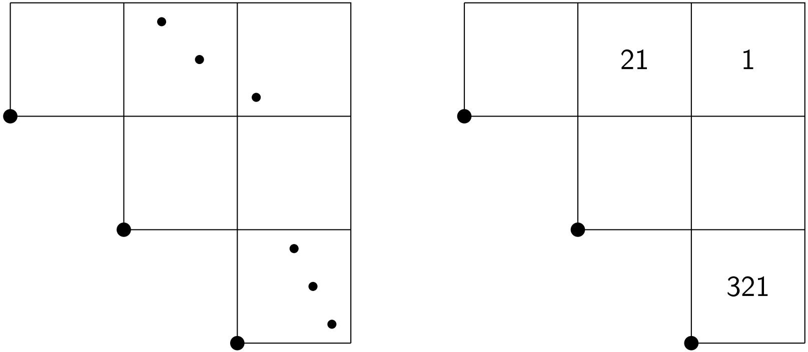
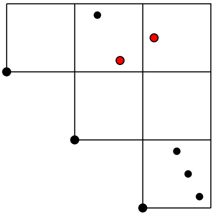


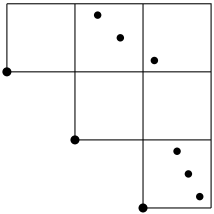
Staircase encoding

Many permutations can have the same staircase encoding

Example



$\underline{6}\,\underline{5}\,9\,\mathbf{7}\,\underline{1}\,\mathbf{8}\,4\,3\,2$



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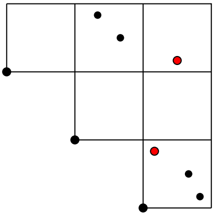

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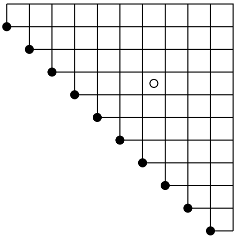
Given a permutation class we need to be able to

- Describe the image of the class under the staircase encoding
- Find the number of permutations in the class that correspond to each staircase encoding in the image, *i.e.*, the number of ways of interleaving rows and columns

Permutations avoiding 123

Avoiding 123 puts constraints on which pairs of cells can contain permutations.



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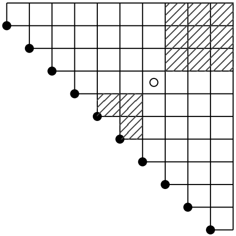

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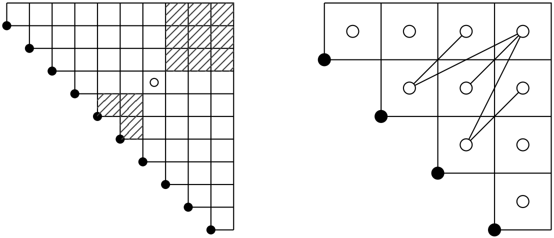

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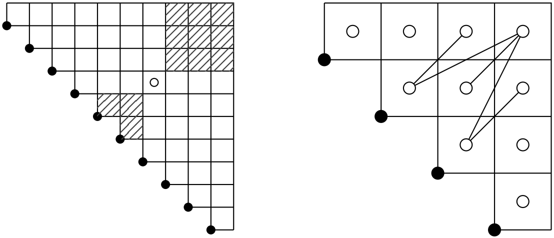

- Encode those restriction by edges
- Non-empty cell of encoding = independent set

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$$F(x, y) = 1 + xF(x, y) + \frac{xyF(x, y)^2}{1 - y(F(x, y) - 1)}.$$

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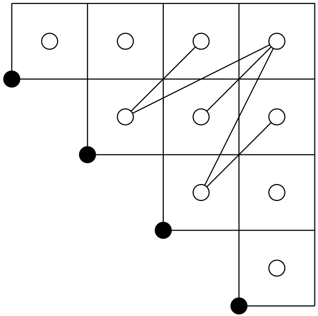
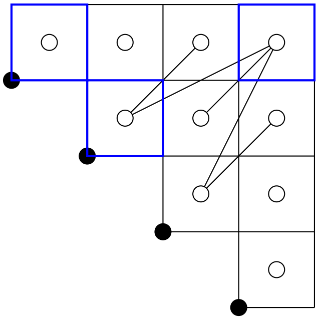
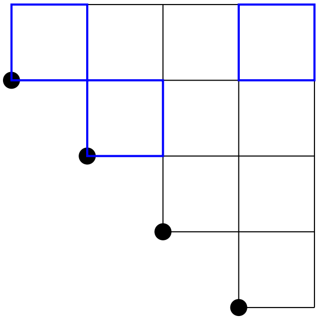
The permutations in all cells of the staircase encoding must avoid 12.





Independent set: x^4y^3



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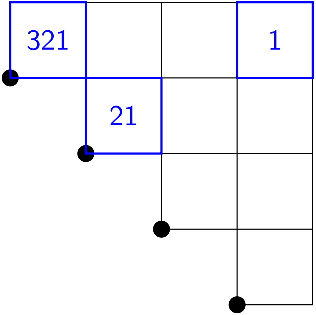

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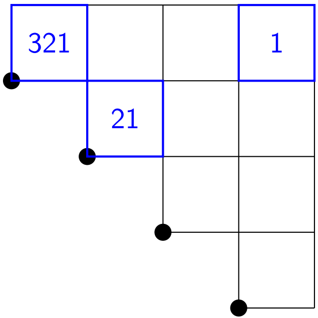

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$F\left(x, \frac{x}{1-x}\right)$ counts staircase encodings of 123 avoiders by size.

Points in two cells in the same row of the grid cannot create 12.

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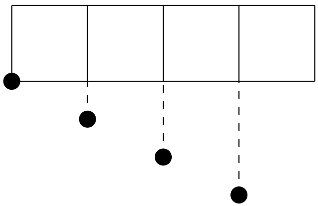
We say that the rows are *decreasing*.



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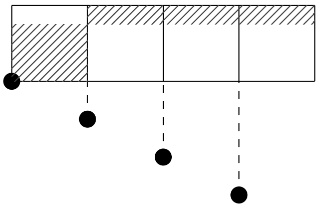

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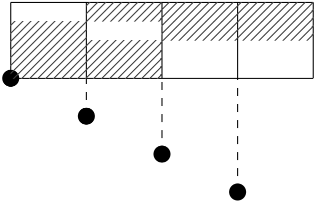

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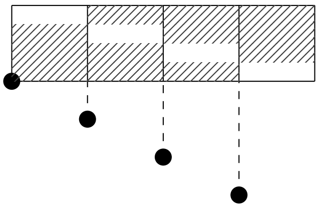

Similarly columns are said to be *decreasing*.
One way of interleaving $\implies$ One 123-avoiders by staircase encoding.

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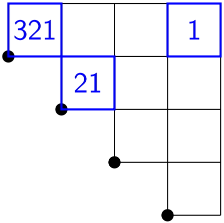

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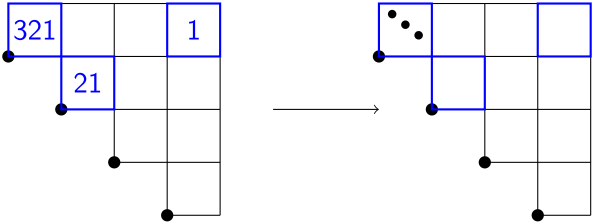

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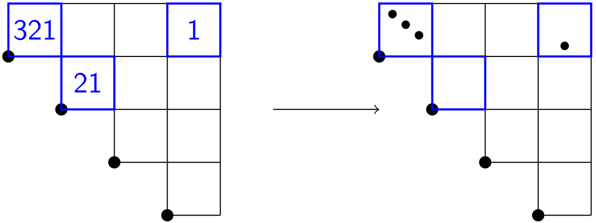

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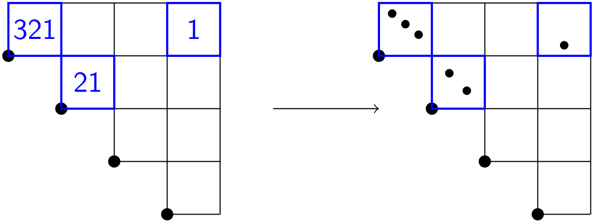

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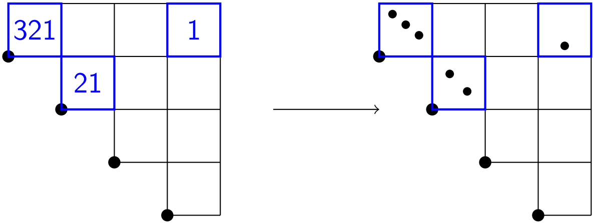
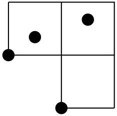


Theorem

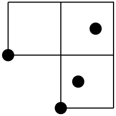
The generating function of Av(123) *is* $F\left(x, \frac{x}{1-x}\right)$.

Avoiding 2314 and 3124

We want to replace 123 with two new patterns:



$r_u = 2314$



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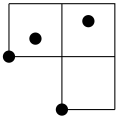

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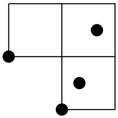

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- Avoiding $r_u \implies$ decreasing rows

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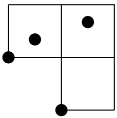

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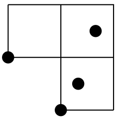

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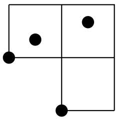

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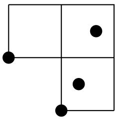

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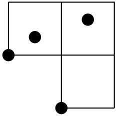

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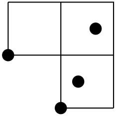

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- Avoiding $c_u \implies$ decreasing columns
- Staircase encoding is a bijection when restricted to $\mathrm{Av}(2314, 3124)$
- Same constraint on the graph for $\mathrm{Av}(123)$

Theorem

The generating function of Av$(2314, 3124$ $)$ *is*

$$F(x, B(x) - 1)$$

where $B(x)$ *is the generating function of* Av$(2314, 3124$ $)$.

Theorem

Let P be a set of skew-indecomposable permutations. The generating function of $\mathrm{Av}(2314, 3124, 1 \oplus P)$ *is*

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Example

$A(x)$, the generation function of $\mathrm{Av}(2314, 3124)$ satisfies

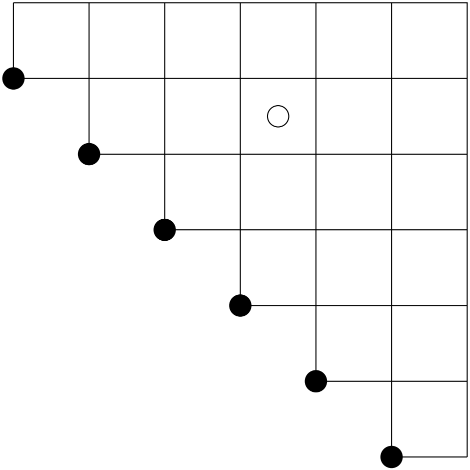
$$A(x) = F(x, A(x) - 1).$$

Solving the equation gives

$$A(x) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}.$$

New cores

We look at avoiders of $r_u = 2314$, $c_u = 3124$ and $c_d = 3142$.



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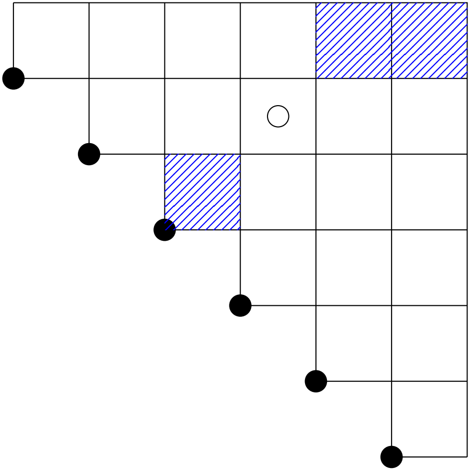

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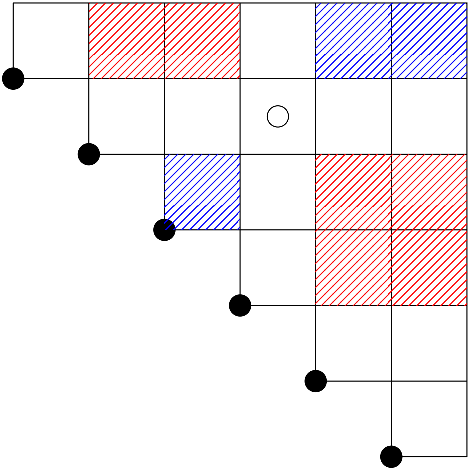

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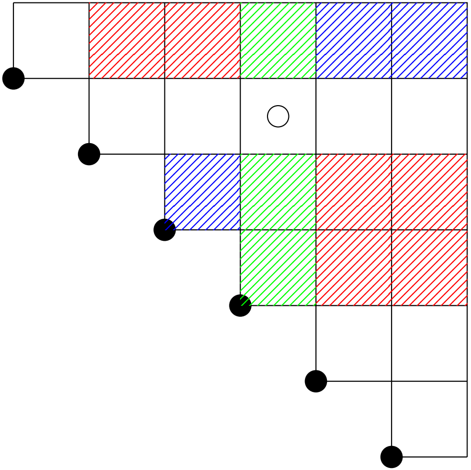

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$$\mathrm{Av}(r_u, c_u, c_d \qquad\qquad) = \mathrm{Av}(2314, 3124, 3142 \qquad\qquad)$$

is

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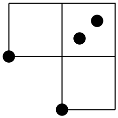
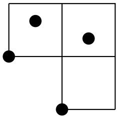
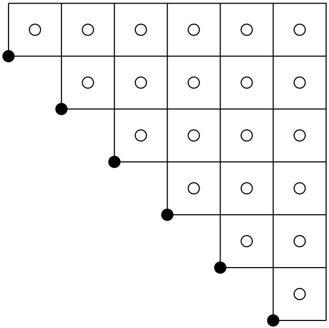
Avoiding 2134 and 2413

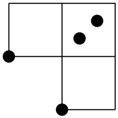


2134

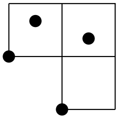


$r_d = 2413$

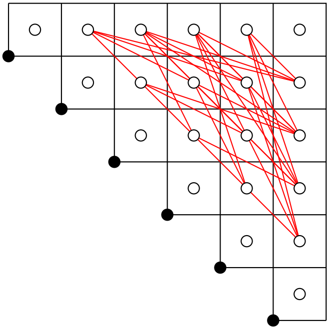


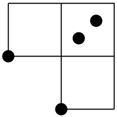


2134

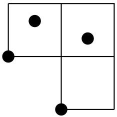


$r_d = 2413$

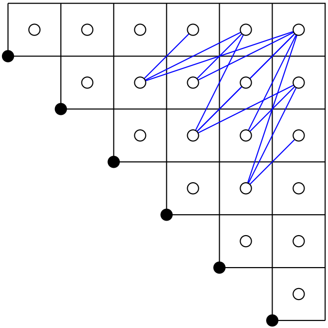


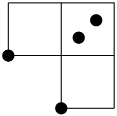


2134

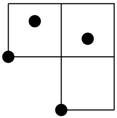


$r_d = 2413$

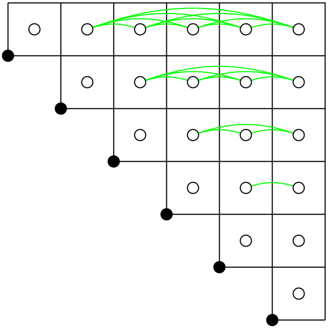


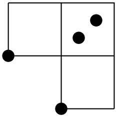


2134

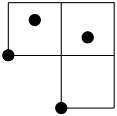


$r_d = 2413$

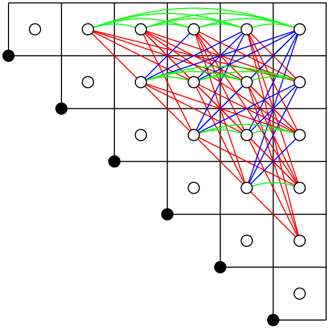


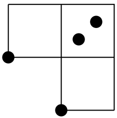


2134

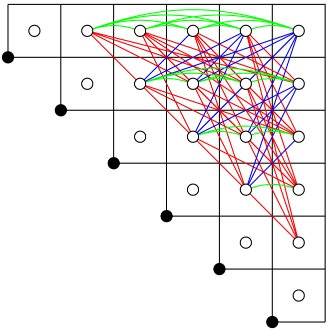


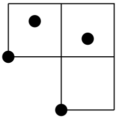
$r_d = 2413$





2134

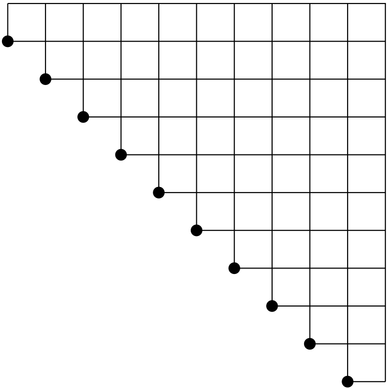




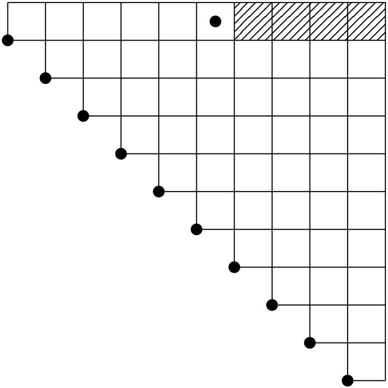
$r_d = 2413$

Remark

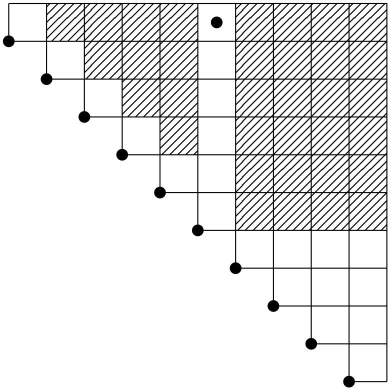
Note that all the diagonal cells are disconnected from the graph.



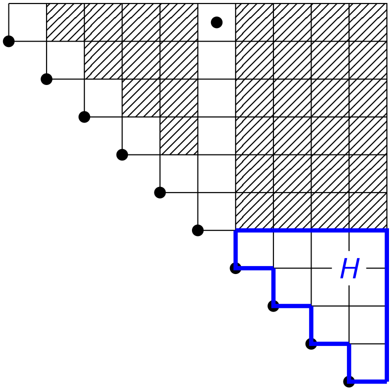
$$H(x,y,z,s) = 1 + xH(x,y,z,s) + \frac{yzsH(x,y,z,s)}{1 - s(y+1)}$$



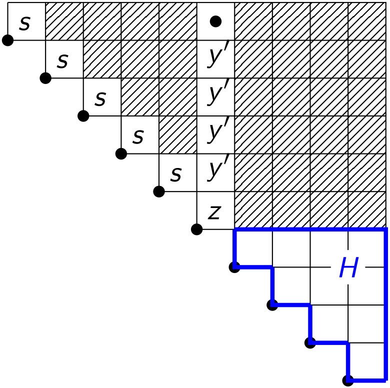
$$H(x,y,z,s) = 1 + xH(x,y,z,s) + \frac{yzsH(x,y,z,s)}{1 - s(y+1)}$$



$$H(x,y,z,s) = 1 + xH(x,y,z,s) + \frac{yzsH(x,y,z,s)}{1 - s(y+1)}$$

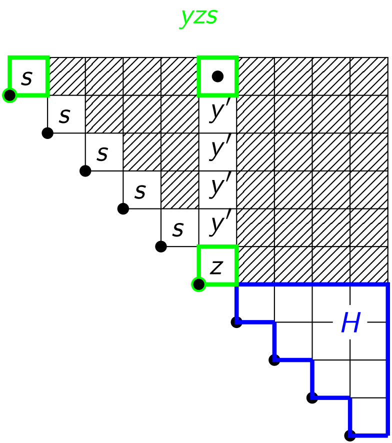


$$H(x,y,z,s) = 1 + xH(x,y,z,s) + \frac{yzsH(x,y,z,s)}{1 - s(y+1)}$$



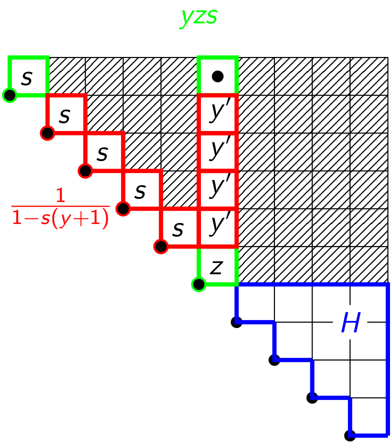
- y for substitution with $\mathrm{Av}^{+}(12)$, $y' = y + 1$
- z for substitution with $\mathrm{Av}^{+}(2413, 2134)$ (with maximum remove)
- s for substitution with $\mathrm{Av}(213)$
- x for substitution with $\mathrm{Av}(2413, 2134)$

$$H(x, y, z, s) = 1 + xH(x, y, z, s) + \frac{yzsH(x, y, z, s)}{1 - s(y + 1)}$$



- y for substitution with $\mathrm{Av}^{+}(12)$, $y' = y + 1$
- z for substitution with $\mathrm{Av}^{+}(2413, 2134)$ (with maximum remove)
- s for substitution with $\mathrm{Av}(213)$
- x for substitution with $\mathrm{Av}(2413, 2134)$

$$H(x, y, z, s) = 1 + xH(x, y, z, s) + \frac{yzsH(x, y, z, s)}{1 - s(y + 1)}$$



- y for substitution with $\mathrm{Av}^+(12)$, $y' = y + 1$
- z for substitution with $\mathrm{Av}^+(2413, 2134)$ (with maximum remove)
- s for substitution with $\mathrm{Av}(213)$
- x for substitution with $\mathrm{Av}(2413, 2134)$

$$H(x,y,z,s) = 1 + xH(x,y,z,s) + \frac{yzsH(x,y,z,s)}{1 - s(y+1)}$$

Theorem

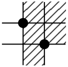
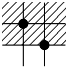
The generating function of Av(2134, 2413) *is*

$$H\left(xB(x), \frac{x}{1-x}, B(x)-1, xC(x)\right)$$

where

- $B(x)$ *is the generating function of* Av(2134, 2413),
- $C(x)$ *is the generating function of* Av(213)

We show that for a set of patterns P satisfying: for all $\pi \in P$

- π is skew-indecomposable,
- π avoids  and
- π contains  or $\pi = \alpha \oplus 1$ with α skew-indecomposable.

Theorem

The generating function of $\operatorname{Av}(2134, 2413, 1 \oplus P)$ *is*

$$H\left(xB(x), \frac{x}{1-x}, B(x)-1, xC(x)\right)$$

where

- $B(x)$ *is the generating function of* $\operatorname{Av}(2134, 2413, {}_{\times}P)$*,*
- $C(x)$ *is the generating function of* $\operatorname{Av}(213, {}_{\times}P^{\times})$

Example

$A(x)$, the generating function of $\mathrm{Av}(2134, 2413)$ satisfies

$$A(x) = H\left(xA(x), \frac{x}{1-x}, A(x) - 1, \frac{1-\sqrt{1-4x}}{2} - 1\right)$$

The equation can be solved explicitly.

Conclusion

Final example

$A(x)$ is the generating function of Av(**2314**, **3124**, 13524, 12435).

$$A(x) = F(x, B(x) - 1)$$

where $B(x)$ is the generating function of Av(**2314**, **3124**, **2413**, 1324).

Final example

$A(x)$ is the generating function of Av(**2314**, **3124**, 13524, 12435).

$$A(x) = F(x, B(x) - 1)$$

where $B(x)$ is the generating function of Av(**2314**, **3124**, **2413**, 1324).

$$B(x) = G(x, C(x) - 1)$$

where $C(x)$ is the generating function of Av(2314, 3124, 2413, 213) = Av(213)

Final example

$A(x)$ is the generating function of Av(**2314**, **3124**, 13524, 12435).

$$A(x) = F(x, B(x) - 1)$$

where $B(x)$ is the generating function of Av(**2314**, **3124**, **2413**, 1324).

$$B(x) = G(x, C(x) - 1)$$

where $C(x)$ is the generating function of Av(2314, 3124, 2413, 213) = Av(213)

Computing $A(x)$ gives the same generating function as for the class Av(2413, 2134).

Basis that can be handled

Basis	Subclasses	References
2314, 3124	8	Schröder number
2413, 3142	8	Schröder number
2314, 3124, 2413, 3142	64	Atkinson & Stitt (2002)
2314, 3124, 2413	8	Mansour & Shattuck (2017)
2314, 3124, 3142*	8	Mansour & Shattuck (2017)
2413, 3142, 2314	8	Callan, Mansour & Shattuck (2017)
2413, 3142, 3124*	8	Callan, Mansour & Shattuck (2017)
2413, 3124	4	Albert, Atkinson & Vatter (2014)
2314, 3142	4	Albert, Atkinson & Vatter (2014)
2134, 2413	2	Albert, Atkinson & Vatter (2014)

*Symmetry of an other class.