Enumeration of Permutation Classes by Inflation of Independent Sets of Graphs

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This talk is based on joint work with Christian Bean and Henning Ulfarsson

We present a way to obtain permutation classes by inflation of independent sets of certain graphs. We cover classes of the form $\mathrm{Av}(2314, 3124, P)$ and $\mathrm{Av}(2413, 3142, P)$. These results allow us to enumerate a total of 48 classes, with bases containing only length 4 patterns. Using a modified approach, we also demonstrate a result for classes of the form $\mathrm{Av}(2134, 2413, P)$ that allows us to enumerate eight more classes described by bases containing only length 4 patterns. We finally use our results to prove an unbalanced Wilf-equivalence between $\mathrm{Av}(2134, 2413)$ and $\mathrm{Av}(2314, 3124, 12435, 13524)$.

Inflating the up-core

Bean, Tannock, and Ulfarsson [1] show a link between the permutations in $\mathrm{Av}(123)$ and independent sets of certain graphs U_n whose vertices are the cells of the staircase grid of size n. These graphs are called up-cores. We extend their results to enumerate $\mathrm{Av}(2314, 3124)$, and moreover certain sub-classes avoiding patterns of the form $1 \oplus \pi$ where π is skew-indecomposable.

More precisely, we choose an independent set of size k in the graph U_n together with a list of k non-empty permutations in $\mathrm{Av}(2314, 3124, P)$ where P is a set of skew-indecomposable permutations. We establish a bijection between these objects and permutations in $\mathrm{Av}(2314, 3124, 1 \oplus P)$. From [1], we get that the number of independent set of size k in an up-core of a staircase grid of size n is given by the coefficient of $x^n y^k$ in the generating function $F(x, y)$ satisfying

$$F = 1 + xF + \frac{xyF^2}{1 - y(F-1)}.$$

Using it, we get the enumeration of these classes:

Theorem 1. *Let P be a set of skew-indecomposable permutations and $A(x)$ be the generating function of* $\mathrm{Av}(2314, 3124, P)$*. Then* $\mathrm{Av}(2314, 3124, 1 \oplus P)$ *is enumerated by $F(x, A-1)$.*

This can be used to enumerate eight classes avoiding length 4 patterns, and many more avoiding longer patterns.

Moreover, using the down-core, also introduced in [1], we state a similar theorem for the class $\mathrm{Av}(2413, 3142, 1 \oplus P)$ where P is a set of sum-indecomposable permutations. This can be used to enumerate eight more classes avoiding length 4 patterns.

Theorem 2. *Let P be a set of sum-indecomposable permutations and $A(x)$ be the generating function of* $\mathrm{Av}(2413, 3142, P)$*. Then* $\mathrm{Av}(2413, 3142, 1 \oplus P)$ *is enumerated by $F(x, A-1)$.*

New cores

We also describe new graphs on the staircase grid of size n. We prove that the number of independent sets of size k for such a graph is given by the coefficient of $x^n y^k$ in the generating function $G(x, y)$ that satisfies

$$G = 1 + xG + \frac{xyG}{1 - x(y+1)}.$$

Using a similar bijection, as we did for Theorem 1, we prove two theorems that enumerate 32 classes avoiding patterns of length 4.

Theorem 3. *Let P be a set of skew-indecomposable permutations and $A(x)$ be the generating function of* $\mathrm{Av}(2314, 3124, 3142, P)$ *(resp.* $\mathrm{Av}(2314, 3124, 2413, P)$*). Then the generating function of* $\mathrm{Av}(2314, 3124, 3142, 1 \oplus P)$ *(resp.* $\mathrm{Av}(2314, 3124, 2413, 1 \oplus P)$*) is $G(x, A-1)$.*

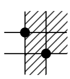
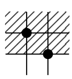
An small modification of G to track the number of rows of the independent set with a third variable also allows to handle classes of the type $\mathrm{Av}(2413, 3142, 3124, 1 \oplus P)$ and $\mathrm{Av}(2413, 3142, 2413, 1 \oplus P)$ for P a set of sum-indecomposable permutations.

Avoiding 2134 and 2413

We use independent sets in a core graph with marked cells in the staircase grid to enumerate of $\mathrm{Av}(2134, 2413)$ and certain sub-classes. The class is symmetric to $\mathrm{Av}(3142, 4312)$ enumerated by Albert, Atkinson, and Vatter [2]. We define

$$_{\times}\pi = \begin{cases} \alpha & \text{if } \pi = 1 \oplus \alpha \\ \pi & \text{otherwise} \end{cases} \qquad \text{and} \qquad \pi^{\times} = \begin{cases} \alpha & \text{if } \pi = \alpha \oplus 1 \\ \pi & \text{otherwise} \end{cases}.$$

We show that for a set of patterns P satisfying that for all $\pi \in P$

- π is skew-indecomposable,
- π avoids  and
- π contains  or $\pi = \alpha \oplus 1$ with α skew-indecomposable.

Theorem 4. *The generating function of* $\mathrm{Av}(2134, 2413, P)$ *is*

$$H\left(xB, \frac{x}{1-x}, B-1, xC\right)$$

where

- *$B(x)$ is the generating function of* $\mathrm{Av}(2134, 2413, {}_{\times}P)$*,*
- *$C(x)$ is the generating function of* $\mathrm{Av}(213, {}_{\times}P^{\times})$*,*

- $H(x,y,z,s) = \frac{s(y+1)-1}{syz+(1-s)x+(1-x)sy+s-1}$

The proofs of all theorems nicely highlight the structure of all the permutation classes. For example, we can extract the structure of the skew-indecomposable permutations in $\mathrm{Av}(2134, 2413)$ as seen in Figure 1.

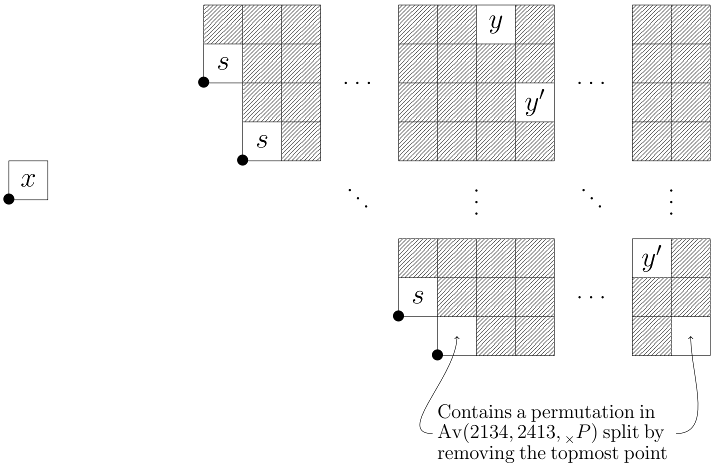


Figure 1: Structure of the two types of skew-indecomposable permutations in $\mathrm{Av}(2134, 2413)$. The cells marked s contain a permutation in $\mathrm{Av}(213)$, the cell marked x contains a permutation in $\mathrm{Av}(2134, 2413)$, the cell marked y contains a non-empty decreasing sequence and the cells marked y' contain a decreasing sequence.

Unbalanced Wilf-equivalence

We demonstrate the Wilf-equivalence of the classes $\mathrm{Av}(2314, 3124, 13524, 12435)$ and $\mathrm{Av}(2134, 2413)$. We first compute $A(x)$, the generating function for the first class using Theorem 1. We get that $A(x) = F(x, B-1)$ where B is the generating function of $\mathrm{Av}(2314, 3124, 2413, 1324)$. Then, with Theorem 3, we get that $B(x) = G(x, C-1)$ where $C(x)$ is the generating function for $\mathrm{Av}(213)$. Since $C(x)$ is known, $A(x)$ can be computed explicitly. Moreover, by Theorem 4, the generating function $D(x)$ of $\mathrm{Av}(2134, 2413)$ satisfies $D(x) = H\left(xD, \frac{x}{1-x}, D-1, xC\right)$. Solving and comparing with $A(x)$ shows the Wilf-equivalence. This fact leads us to believe that a direct proof using the core structure might be possible.

Several other unbalanced Wilf-equivalences can be derived using our theorems.

References

[1] Christian Bean, Murray Tannock and Henning Ulfarsson. *Pattern avoiding permutations and independent sets in graphs.* arXiv:1512.08155 (2015). Submitted.

[2] Michael H. Albert, M. D. Atkinson, and Vincent Vatter. *Inflations of geometric grid classes: three case studies.* Australasian Journal of Combinatorics 58.1 (2014): 27-47.