Enumeration of Permutation Classes by Inflation of Independent Sets of Graphs

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(based on joint work with Christian Bean and Henning Ulfarsson)

Reykjavik University

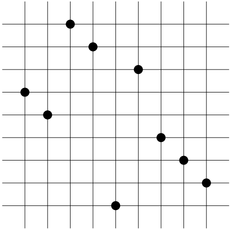
Permutations Patterns 2019

Staircase encoding

For any permutation π we can extract the left-to-right minima and place them on the diagonal of a square grid.

Example

$\pi = \underline{6}\underline{5}98\underline{1}7432$



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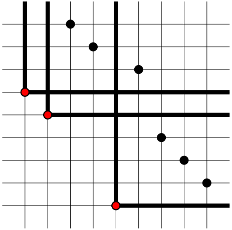

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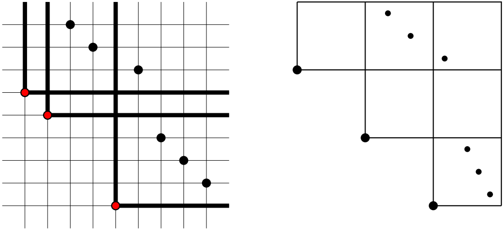

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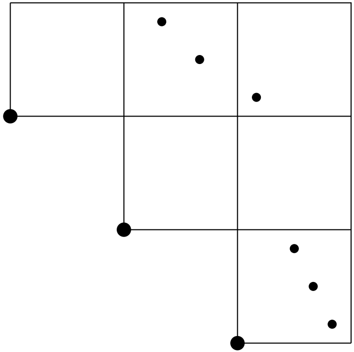

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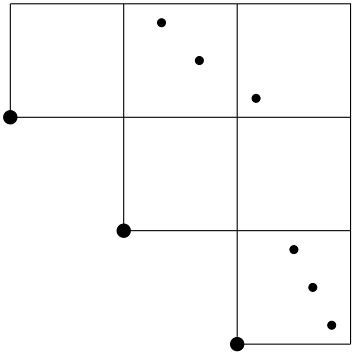
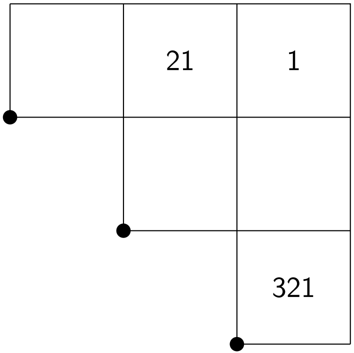




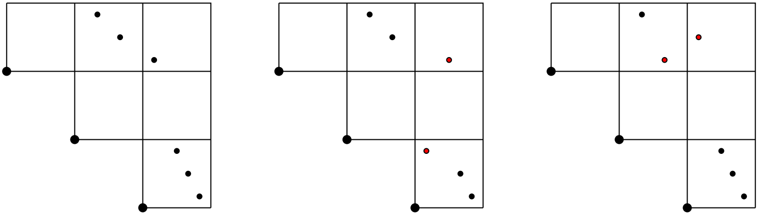
Staircase encoding

Many different permutations can have the same staircase encoding

Example

$$\pi' = \underline{6}\,\underline{5}\,98\,\underline{1}\,\mathbf{4}3\mathbf{7}2 \qquad \text{and} \qquad \pi'' = \underline{6}\,\underline{5}\,9\mathbf{7}\,\underline{1}\,\mathbf{8}432$$

have the same staircase encoding has the permutation π.



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Given a permutation class we need to be able to

- Describe the image of the class under the staircase encoding
- Find the number of permutations in the class that correspond to each staircase encoding in the image, *i.e.*, the number of ways of interleaving rows and columns

Permutations avoiding 123

123 avoiders

We start with the example of Av(123) from Bean, Tannock and Ulfarsson in *Pattern avoiding permutations and independent sets in graphs*.

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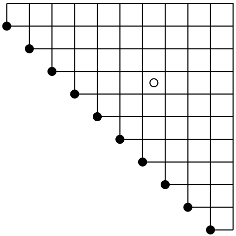
To describe all the staircase encodings that can be obtained from 123 avoiders we follow a two step process

1. Find all the possible sets of active cells for a staircase encoding
2. Find the permutations that can occupy any of those cells

Sets of active cells

Avoiding 123 puts constraints on which pairs of cells can contain permutations.



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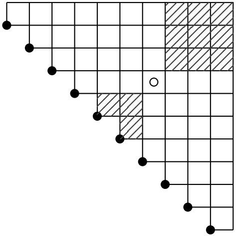

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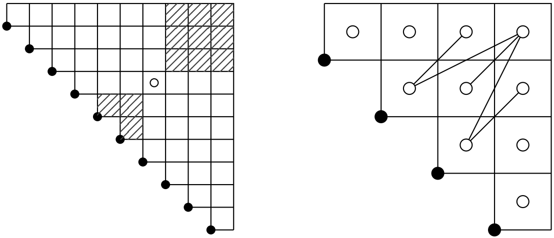

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$$F(x, y) = 1 + xF(x, y) + \frac{xyF(x, y)^2}{1 - y(F(x, y) - 1)}.$$

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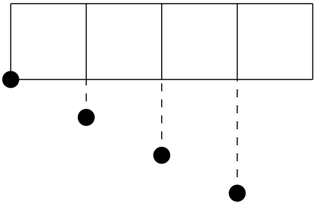
$$F(x, y) = 1 + xF(x, y) + \frac{xyF(x, y)^2}{1 - y(F(x, y) - 1)}.$$

Finally, the permutations in all cells of the staircase encoding must avoid 12.

From encoding to permutations

Points in two cells in the same row of the grid cannot create 12. Hence, all points of the left cell are above the points of the right cell. We say that the rows are *decreasing*.



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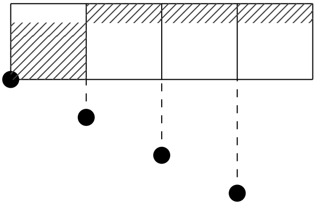

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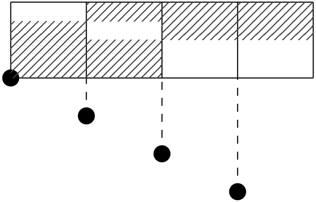

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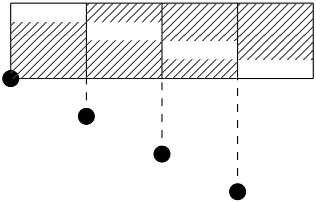

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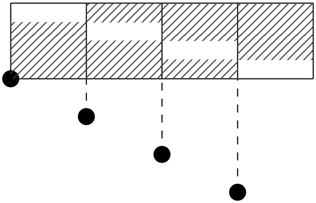

Similarly columns are said to be *decreasing*.
For each staircase encoding, only one permutation in Av(123) is mapped to it by the staircase encoding because only one interleaving is possible.

The number of staircase encodings for 123 avoiders of length n is given by the generating function $F\left(x, \frac{x}{1-x}\right)$.
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Theorem

The generating function of Av(123) *is* $F\left(x, \frac{x}{1-x}\right)$.

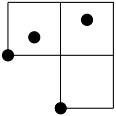
Remark

A symmetric results can be stated for 132 avoiders.

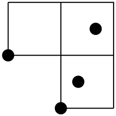
Avoiding 2314 and 3124

Row-up and column-up patterns

We want to replace 123 with two new patterns: $r_u = 2314$ and $c_u = 3124$.



r_u

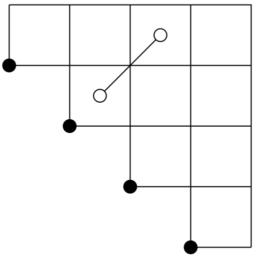


c_u

We use this notation since r_u forbids increasing sequences along rows while c_u forbids increasing sequences along columns. Hence, all permutations in $\mathrm{Av}(2314, 3124)$ have a different staircase encoding.

Sets of active cells

If we look at the left-to-right minima of the patterns as left-to-right minima on the grid we see the same constraints for sets of active cells as in the 123 case.



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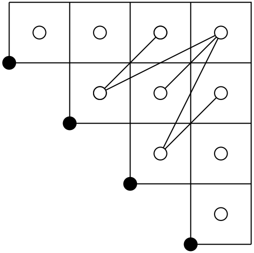

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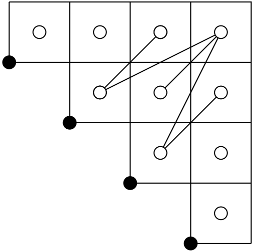

$F(x, y)$ also describes the independent sets.

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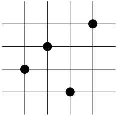
Each staircase encoding gives a permutation in the class



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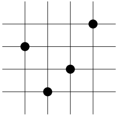

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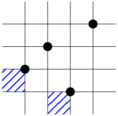

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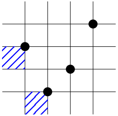

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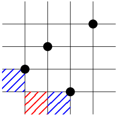

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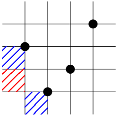

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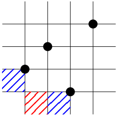

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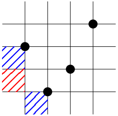

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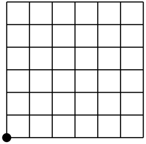

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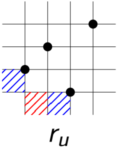

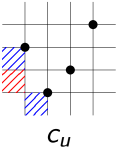

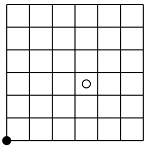

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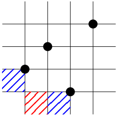

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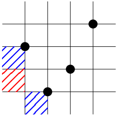

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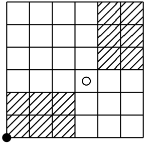

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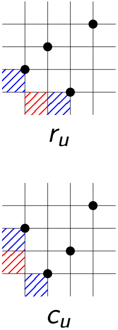

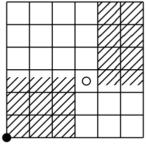

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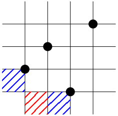

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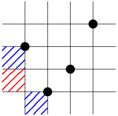

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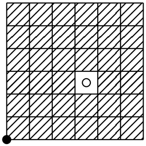

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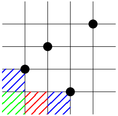

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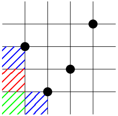

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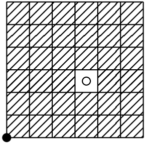


Enumeration

Theorem

Let P be a set of skew-indecomposable permutations. The generating function of $\mathrm{Av}(2314, 3124, 1 \oplus P)$ *is*

$$F(x, B(x) - 1)$$

where $B(x)$ is the generating function of $\mathrm{Av}(2314, 3124, P)$.

Example

The generating function of $\mathrm{Av}(2314, 3124, 1234)$ is

$$F\left(x, \frac{1 - \sqrt{1 - 4x}}{2x} - 1\right)$$

since $\mathrm{Av}(2314, 3124, 123) = \mathrm{Av}(123)$.

Enumeration

Example

$A(x)$, the generation function of Av$(2314, 3124)$ satisfies

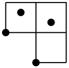
$$A(x) = F(x, A(x) - 1).$$

Solving the equation gives

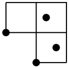
$$A(x) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}.$$

Remark

A symmetric results can be derived for $r_d = 2143$ and $c_d = 3142$ and sum-indecomposable patterns.



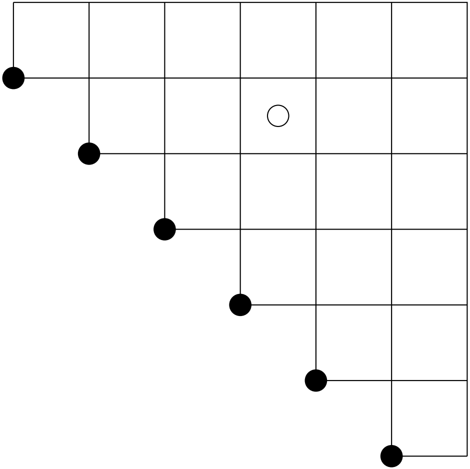
r_d



c_d

New cores

We look at avoiders of $r_u = 2314$, $c_u = 3124$ and $c_d = 3142$.



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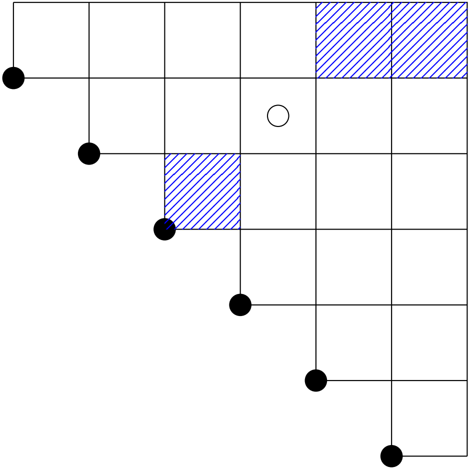

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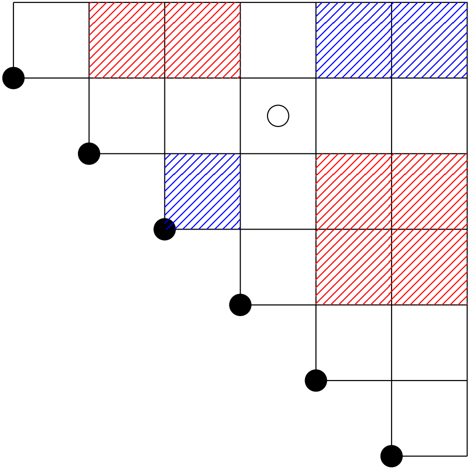

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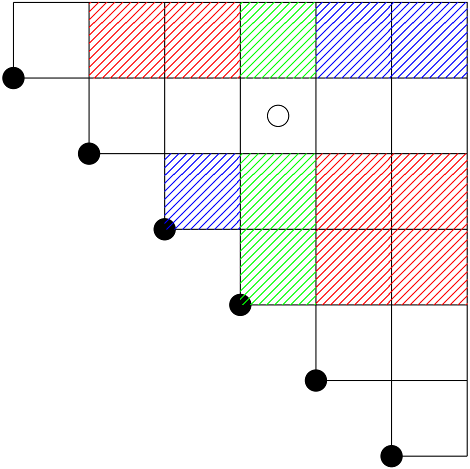

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$$\mathrm{Av}(r_u, c_u, c_d, 1 \oplus P) = \mathrm{Av}(2314, 3124, 3142, 1 \oplus P)$$

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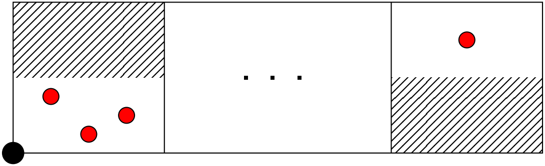
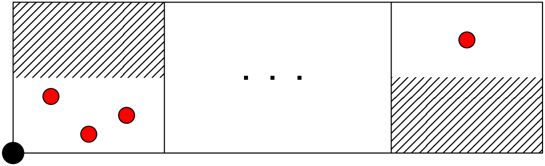
Remark

A symmetric version can be done for r_u, c_u, c_d.

Regarding the bases $\mathrm{Av}(r_d, c_d, c_u)$ and $\mathrm{Av}(r_d, c_d, r_u)$ some extra care is needed since c_u and r_u are sum-indecomposable.



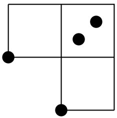
Regarding the bases $\mathrm{Av}(r_d, c_d, c_u)$ and $\mathrm{Av}(r_d, c_d, r_u)$ some extra care is needed since c_u and r_u are sum-indecomposable.



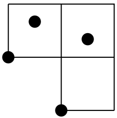
However this can be handled by

- tracking the number of rows/columns of the independent set
- using a different permutation class for the leftmost/topmost cell in each row

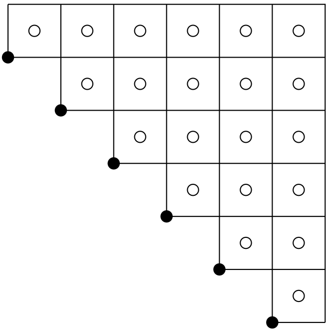
Avoiding 2134 and 2413

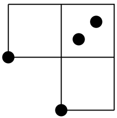


2134

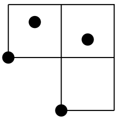


$r_d = 2413$

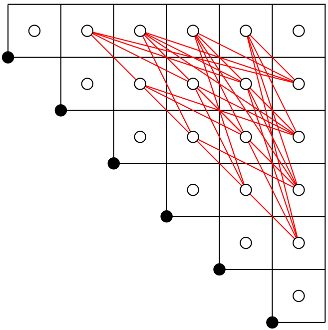


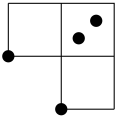


2134

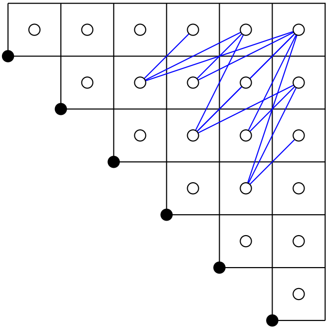


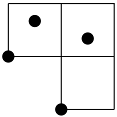
$r_d = 2413$





2134





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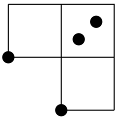
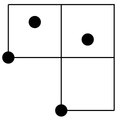
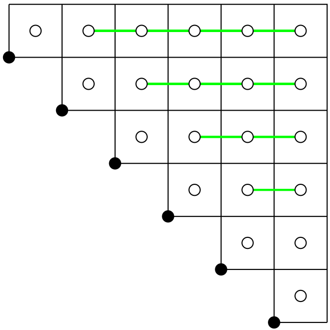
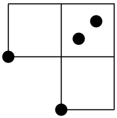


2134



$r_d = 2413$





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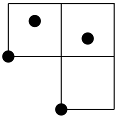
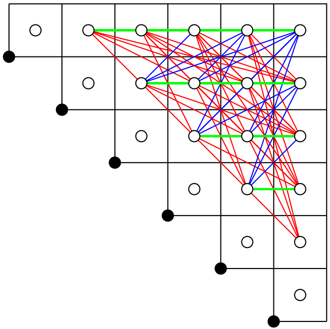
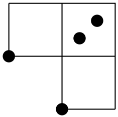
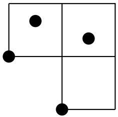


$r_d = 2413$

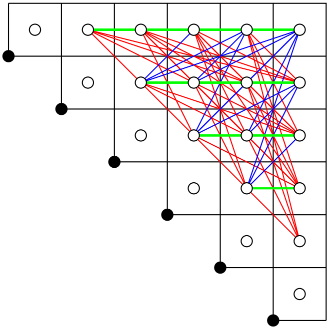




2134

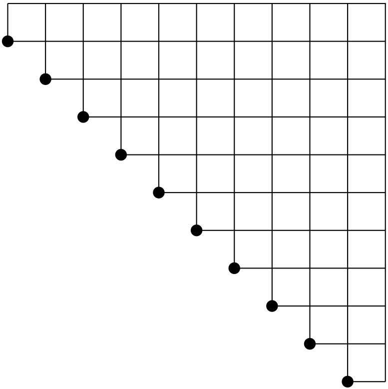


$r_d = 2413$

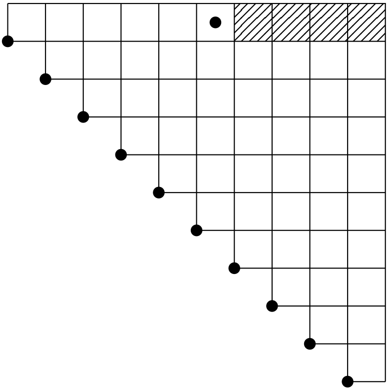


Remark

Note that all the diagonal cells are disconnected from the graph.



$$H(x,y,z,s) = 1 + xH(x,y,z,s) + \frac{yzsH(x,y,z,s)}{1 - s(y+1)}$$



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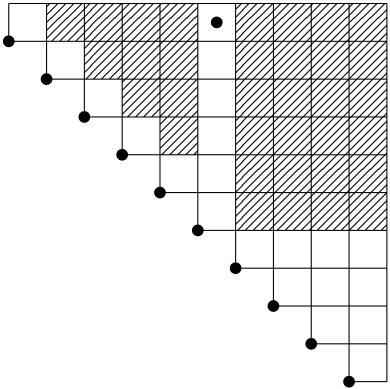

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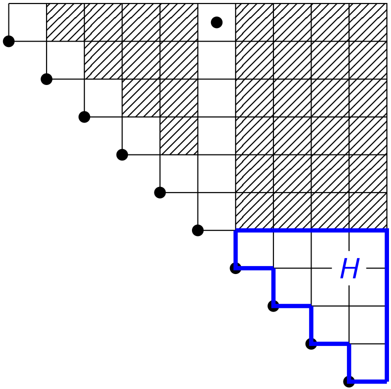

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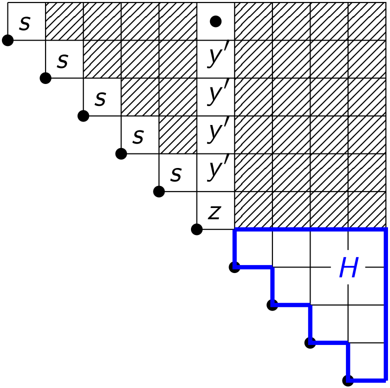
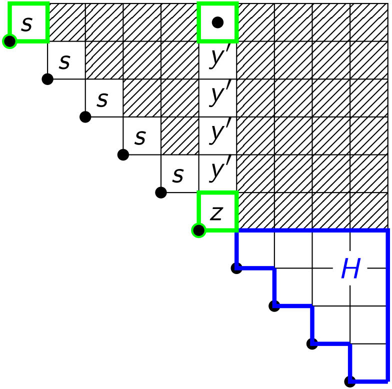


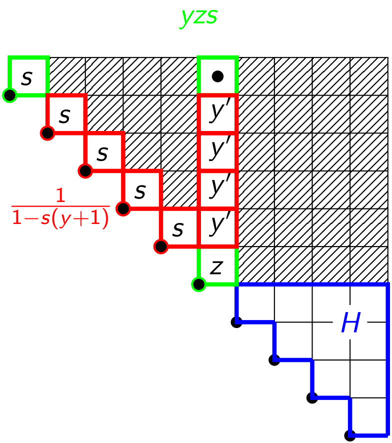
- x for substitution with Av(2413, 2134)
- y for substitution with $\mathrm{Av}^{+}(12)$, $y' = y + 1$
- z for substitution with $\mathrm{Av}^{+}(2413, 2134)$ (with maximum remove)
- s for substitution with Av(213)

$$H(x, y, z, s) = 1 + xH(x, y, z, s) + \frac{yzsH(x, y, z, s)}{1 - s(y + 1)}$$



- x for substitution with Av$(2413, 2134)$
- y for substitution with Av$^{+}(12)$, $y' = y + 1$
- z for substitution with Av$^{+}(2413, 2134)$ (with maximum remove)
- s for substitution with Av(213)

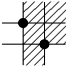
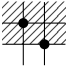
$$H(x, y, z, s) = 1 + xH(x, y, z, s) + \frac{yzsH(x, y, z, s)}{1 - s(y + 1)}$$



- x for substitution with Av(2413, 2134)
- y for substitution with $\mathrm{Av}^+(12)$, $y' = y + 1$
- z for substitution with $\mathrm{Av}^+(2413, 2134)$ (with maximum remove)
- s for substitution with Av(213)

$$H(x, y, z, s) = 1 + xH(x, y, z, s) + \frac{yzsH(x, y, z, s)}{1 - s(y + 1)}$$

We show that for a set of patterns P satisfying: for all $\pi \in P$

- π is skew-indecomposable,
- π avoids  and
- π contains  or $\pi = \alpha \oplus 1$ with α skew-indecomposable.

Theorem

The generating function of $\mathrm{Av}(2134, 2413, 1 \oplus P)$ *is*

$$H\left(xB, \frac{x}{1-x}, B-1, xC\right)$$

where

- $B(x)$ *is the generating function of* $\mathrm{Av}(2134, 2413, {}_{\times}P)$,
- $C(x)$ *is the generating function of* $\mathrm{Av}(213, {}_{\times}P^{\times})$,

Example

$A(x)$, the generating function of $\mathrm{Av}(2134, 2413)$ satisfies

$$A(x) = H\left(xA(x), \frac{x}{1-x}, A(x) - 1, \frac{1-\sqrt{1-4x}}{2} - 1\right)$$

The equation can be solved explicitly.

Unbalanced Wilf-equivalence

$A(x)$ is the generating function of Av(**2314**, **3124**, 13524, 12435).

$$A(x) = F(x, B(x) - 1)$$

where $B(x)$ is the generating function of Av(**2314**, **3124**, **2413**, 1324).

Unbalanced Wilf-equivalence

$A(x)$ is the generating function of Av(**2314**, **3124**, 13524, 12435).

$$A(x) = F(x, B(x) - 1)$$

where $B(x)$ is the generating function of Av(**2314**, **3124**, **2413**, 1324).

$$B(x) = G(x, C(x) - 1)$$

where $C(x)$ is the generating function of Av(2314, 3124, 2413, 213) = Av(213)

Unbalanced Wilf-equivalence

$A(x)$ is the generating function of Av(**2314**, **3124**, 13524, 12435).

$$A(x) = F(x, B(x) - 1)$$

where $B(x)$ is the generating function of Av(**2314**, **3124**, **2413**, 1324).

$$B(x) = G(x, C(x) - 1)$$

where $C(x)$ is the generating function of Av(2314, 3124, 2413, 213) = Av(213)
Computing $A(x)$ gives the same generating function as for the class Av(2413, 2134).

Conclusion

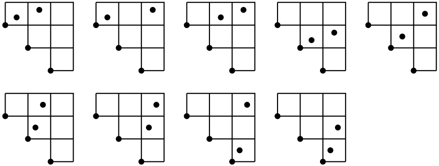
Basis that can be handled

Basis	Subclasses	References
2314, 3124	8	Schröder number
2413, 3142	8	Schröder number
2314, 3124, 2413, 3142	64	Atkinson & Stitt (2002)
2314, 3124, 2413	8	Mansour & Shattuck (2017)
2314, 3124, 3142*	8	Mansour & Shattuck (2017)
2413, 3142, 2314	8	Callan, Mansour & Shattuck (2017)
2413, 3142, 3124*	8	Callan, Mansour & Shattuck (2017)
2413, 3124	4	Albert, Atkinson & Vatter (2014)
2314, 3142	4	Albert, Atkinson & Vatter (2014)
2134, 2413	2	Albert, Atkinson & Vatter (2014)

*Symmetry of an other class.

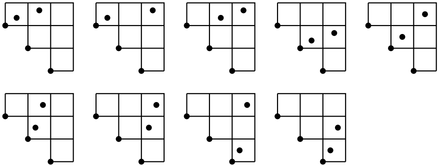
Future work

- Using length five patterns with 3 left-to-right minima

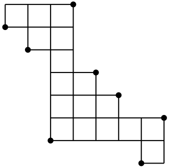


Future work

- Using length five patterns with 3 left-to-right minima

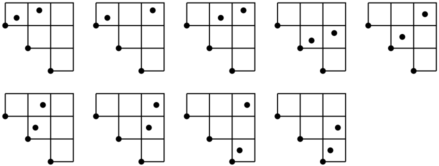


- Consider also the right-to-left maxima



Future work

- Using length five patterns with 3 left-to-right minima



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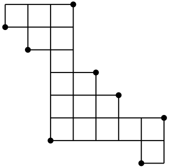

- Other Wilf-equivalences and bijective proof