Occurrence graphs of patterns in permutations
(ísl. *Tilvikanet mynstra í umröðunum*)

Stærðfræði á Íslandi 2015

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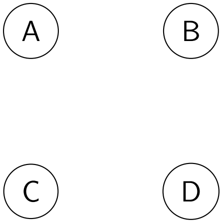

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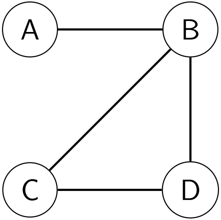

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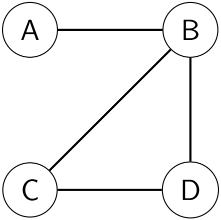

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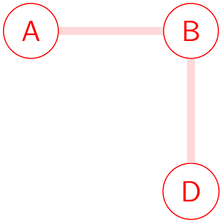

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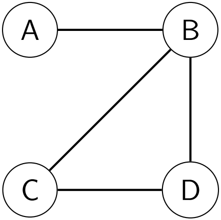

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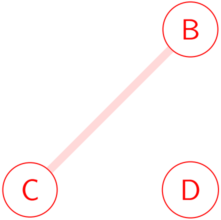

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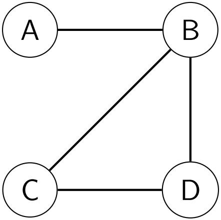

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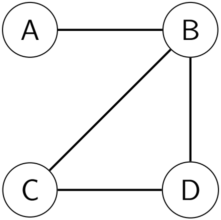

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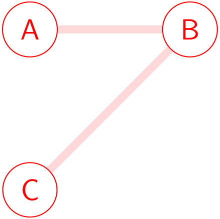

Mynd: *The induced subgraph of the vertices A,B,C*

Definition of Permutations

- We let $[\![1, n]\!]$ denote the integer interval $\{1, \ldots, n\}$.

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- We denote the permutation with $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$.
- The *set of permutations* (ísl. *mengi allra umraðana*) of length n is denoted by $\mathfrak{S}_n$. The set of all permutations is $\mathfrak{S} = \cup_{n=0}^{+\infty}\mathfrak{S}_n$.

Grid plot

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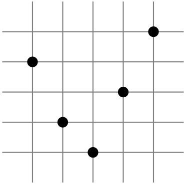
$\mathrm{Grid}(42135) =$ 

Pattern standardisation

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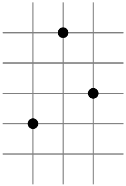

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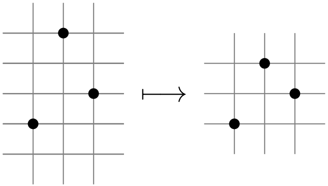

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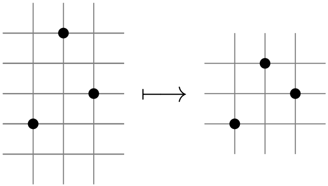

For example $\mathrm{st}(253) = 132$ and $\mathrm{st}(132) = 132$.

Pattern containment and occurrences

We say that a permutation π *contains* (ísl. *inniheldur*) a smaller permutation p (called a *(classical permutation) pattern*) (ísl. (*klassískt umröðunar-*)*mynstur*) if there exists a substring of π such that the standardisation of it is equal to p.

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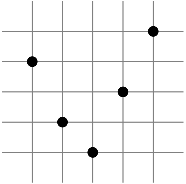

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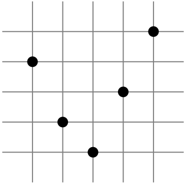

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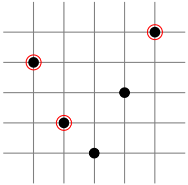

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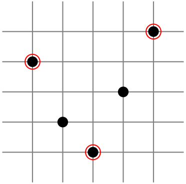

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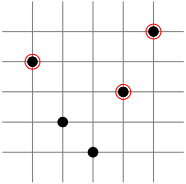

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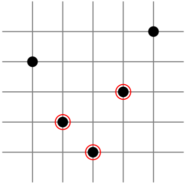

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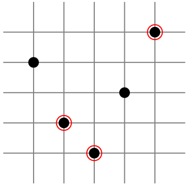

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Pattern containment and occurrences (continued)

- The substrings 425, 415, 435, 213 and 215 are the *occurrences* (ísl. *tilvik*) of 213 in 42135.

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- The corresponding *index sets* (ísl. *vísamengi*) are $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$.
- The set of all index sets of $p = 213$ in $\pi = 42135$ is the *occurrence set* (ísl. *tilvikamengi*) of p in π, denoted with $V_p(\pi)$.

Definition of occurence graphs

For a pattern $p \in \mathfrak{S}_k$ and for a permutation $\pi \in \mathfrak{S}_n$ we define the *occurrence graph* (ísl. *tilvikanet*) $G_p(\pi)$ of p in π in the following way:

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- The set of vertices is $V_p(\pi)$, the occurrence set of p in π.
- uv is an edge in $G_p(\pi)$ if the vertices $u = \{u_1, \ldots, u_k\}$ and $v = \{v_1, \ldots, v_k\}$ in $V_p(\pi)$ differ by exactly one element, i.e. if

$$|u \setminus v| = |v \setminus u| = 1.$$

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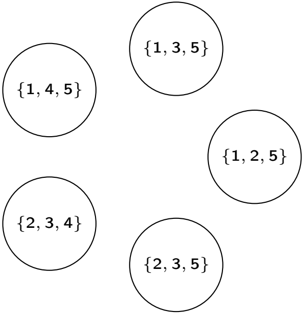

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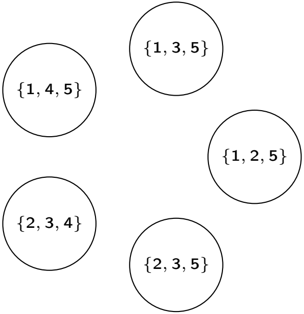
In previous example we derived the occurrence set $V_{213}(42135)$. We compute the edges of $G_{213}(42135)$ by comparing the vertices two at a time to see if the sets differ by exactly one element. The graph is shown below:



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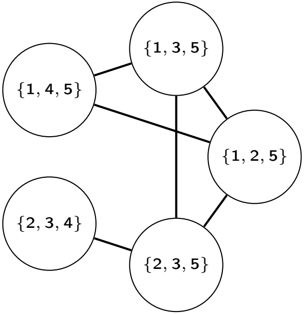

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Let c be a property of graphs and let

$$\mathscr{G}_{p,c} = \{\pi \in \mathfrak{S} \colon\ G_p(\pi) \text{ has property } c\}.$$

Main theorem

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Theorem

Let c be a hereditary property of graphs. For any pattern p the set $\mathscr{G}_{p,c}$ is a permutation class, i.e. there is a set of classical permutations patterns M such that

$$\mathscr{G}_{p,c} = \mathrm{Av}(M).$$

Forests and trees

Theorem

Let c be the property of being a forest and $p = 12$. Then

$$\mathscr{G}_{p,c} = \mathrm{Av}\left(123, 1432, 2143, 3214\right).$$

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The number of permutations of length n in $\mathscr{G}_{12,\mathrm{tree}}$ is $(n-1)^2$.