Occurrence graphs of patterns in permutations

Permutation Patterns 2016

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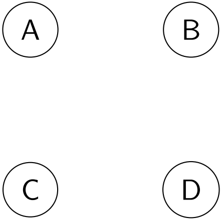

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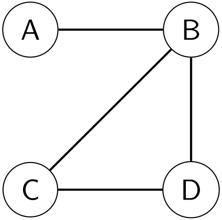

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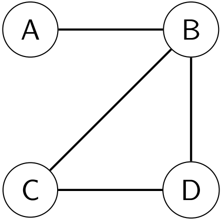

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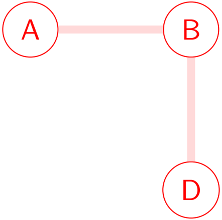

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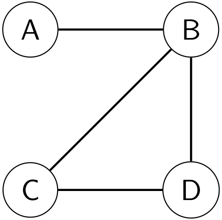

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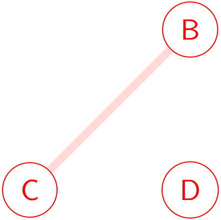

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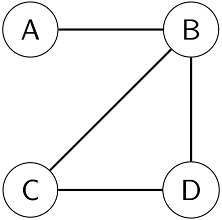

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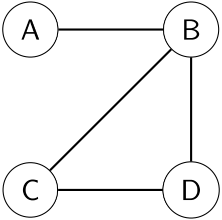

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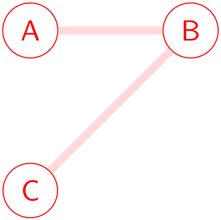

Figure: The induced subgraph of the vertices A, B, C

Grid plot

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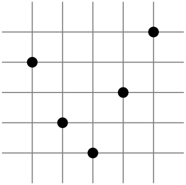

Pattern containment and occurrences

We say that a permutation π *contains* a smaller permutation p (called a (*classical permutation*) *pattern*) if there exists a substring of π such that the standardisation of it is equal to p

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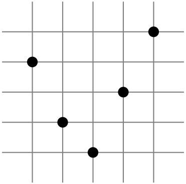

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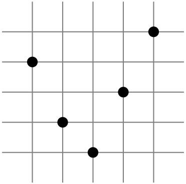

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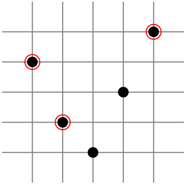

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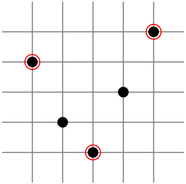

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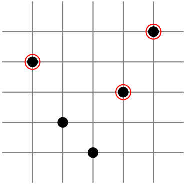

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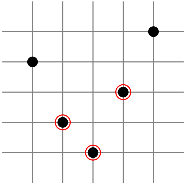

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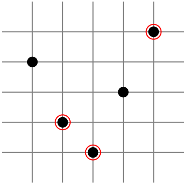

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- The substrings 425, 415, 435, 213 and 215 are the *occurrences* of 213 in 42135

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- The corresponding *index sets* are $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$
- The set of all index sets of $p = 213$ in $\pi = 42135$ is the *occurrence set* of p in π, denoted with $V_p(\pi)$

Definition of occurence graphs

For a pattern $p \in \mathfrak{S}_k$ and for a permutation $\pi \in \mathfrak{S}_n$ we define the *occurrence graph* $G_p(\pi)$ of p in π in the following way:

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- The set of vertices is $V_p(\pi)$, the occurrence set of p in π
- uv is an edge in $G_p(\pi)$ if the vertices $u = \{u_1, \dots, u_k\}$ and $v = \{v_1, \dots, v_k\}$ in $V_p(\pi)$ differ by exactly one element, i.e. if

$$|u \setminus v| = |v \setminus u| = 1$$

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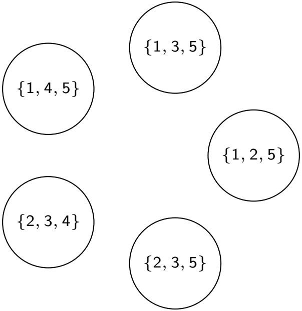

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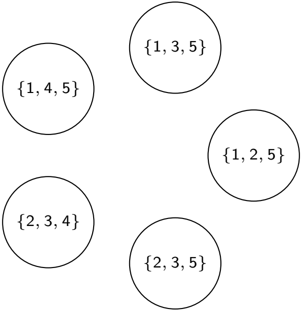

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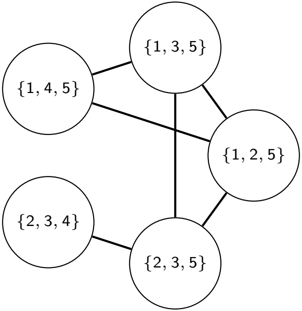

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because such a clique can be formed by choosing an index i among $1, \ldots, n$ to be the shared index in all the vertices in the clique. The remaining indices can be chosen from $\{1, \ldots, n\} \setminus i$

Hereditary property

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Let c be a property of graphs and let

$$\mathscr{G}_{p,c} = \{\pi \in \mathfrak{S} \colon G_p(\pi) \text{ has property } c\}$$

Main theorem

Theorem

Let c be a hereditary property of graphs. For any pattern p the set $\mathscr{G}_{p,c}$ is a permutation class, i.e. there is a set of classical permutation patterns M such that

$$\mathscr{G}_{p,c} = \mathrm{Av}(M)$$

Bipartite graphs

Table: Experiments for bipartite occurrence graphs. (Up to length 8)

p	basis	Number seq.
12	123, 1432, 3214	$1, 2, 5, 12, 26, 58, 131, 295$
123	1234, 12543, 14325, 32145	$1, 2, 6, 23, 100, 462, 2207, 10758$
132	1432, 12354, 13254, 13452, 15234, 21354, 23154, 31254, 32154	$1, 2, 6, 23, 95, 394, 1679, 7358$

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The first line follows from the fact that every cycle in $G_{12}(\pi)$ has length 3 or 4. The sequence is on OEIS. The others are not

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Table: Experiments for occurrence graphs that are forests. (Up to length 8)

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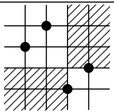
connected		$1, 2, 6, 23, 111, 660, 4656$
tree	infinite non-classical basis	$0, 1, 4, 9, 16, 25, 36, 49$
chordal	1234, 1243, 1324, 2134, 2143	$1, 2, 6, 19, 61, 196, 630, 2025$
clique	1234, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2413, 3124, 3142, 3412	$1, 2, 6, 12, 20, 30, 42, 56$

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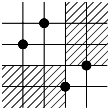
connected		$1, 2, 6, 23, 111, 660, 4656$
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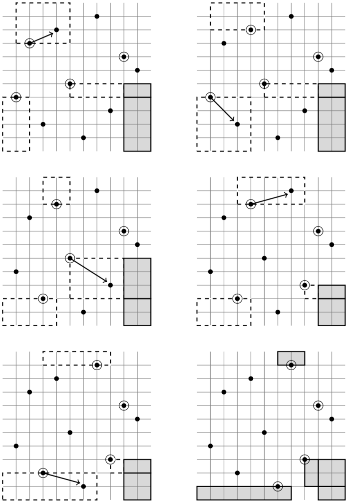
The first line follows by staring. Also easy to count tree result

Motivation



Crushed hopes

Pattern statistics are hard: How many permutations of length n have exactly k occurrence of the pattern p?

The end

Questions, comments?