Pattern Avoidance and Non-Crossing Subgraphs of Polygons

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Definition (Permutation)

A *permutation* is considered to be an arrangement of the numbers $1, 2, \ldots, n$ for some positive n.

Definition (Pattern)

A permutation, or *pattern*, π is said to be contained in, or be a *subpermutation* of, another permutation, σ if σ contains a subsequence order isomorphic to π.

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$$\pi = 314592687$$
$$\sigma = 1423$$

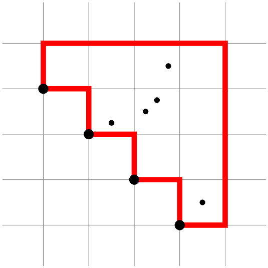
Definition (Classical Permutation Classes)

A Classical *permutation class*, is a set of permutations closed downwards under the subpermutation relation. We define a classical permutation class by stating the minimal set of permutations that it avoids.

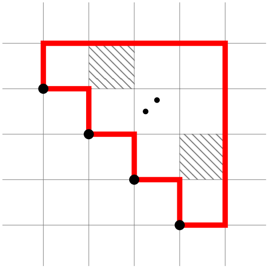
This minimal forbidden set of patterns is known as the *basis* for the class. The class with basis B is denoted $Av(B)$ and $Av_n(B)$ is used to denote the set of permutations of length n in $Av(B)$.

132-avoiding permutations

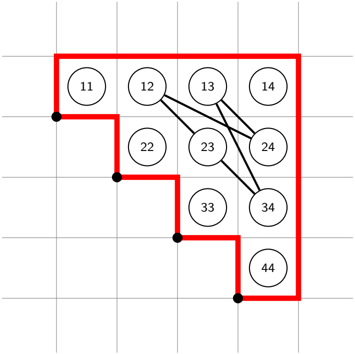
Given any permutation, π, we can extract the left-to-right minima.



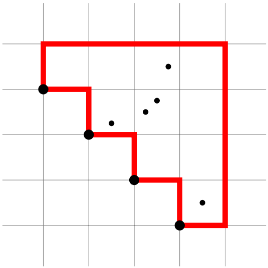
For any permutation $\pi \in Av_n(132)$



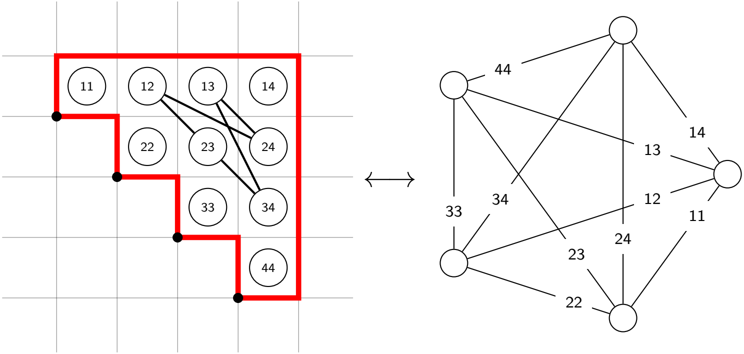
Given this representation we can construct a graph



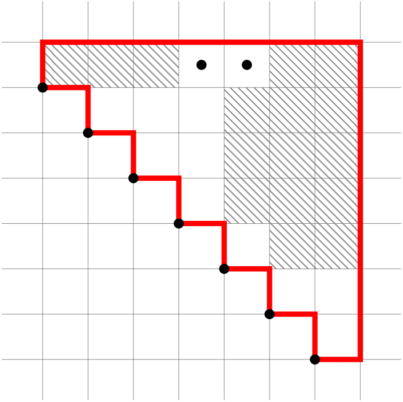
An independent set of size k and a positive integer sequence of length k uniquely determines a 132-avoider.

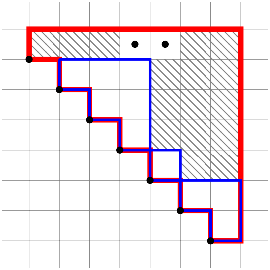


There is in bijection with non-crossing subgraphs on a regular polygon and the independent sets.



We can also directly enumerate this.





d	F
0	$x \cdot F$
1	$x \cdot y \cdot F^2$
2	$x \cdot y^2 \cdot F^2(F-1)$
$\vdots$	$\vdots$
n	$x \cdot y^n \cdot F^2(F-1)^{n-1}$
$\vdots$	$\vdots$

Deriving the generating function.

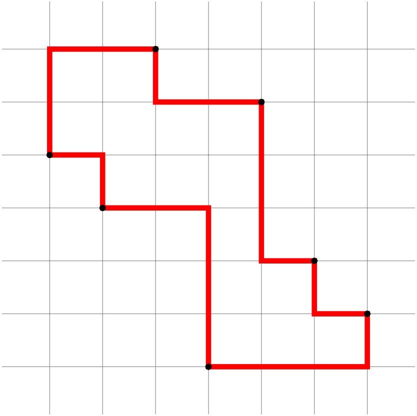
This leads to the following generating function:

$$F(x,y) = 1 + x \cdot F(x,y) + \frac{xy \cdot F(x,y)^2}{1 - y \cdot (F(x,y) - 1)}.$$

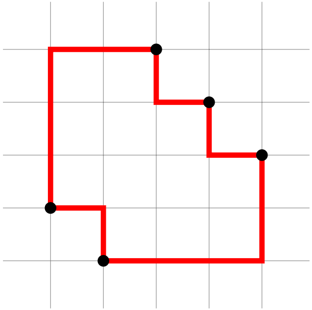
Evaluating $F\left(x, \frac{x}{1-x}\right)$ gives the Catalan numbers.

1324-avoiders

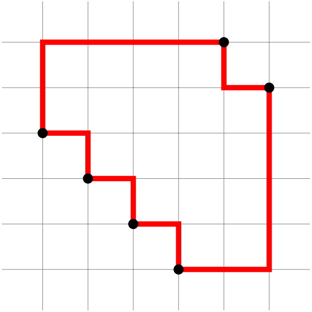
Given $\pi \in Av_n(1324)$ we can extract the boundary.



Non-intersecting boundary of a 1324-avoider.



For 1324-avoiders with non-intersecting boundary and two right to left maxima.



Let:

$$G = \frac{x^2 \cdot F}{1 - y \cdot (F - 1)}$$

Then:

$$H = \frac{G + 1}{1 - y \cdot G} - 1$$

This corresponds to non-crossing subgraphs on a polygon with multiple edges as shown below.