Pattern Avoidance and Non-Crossing Subgraphs of Polygons

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We illustrate a connection between independent sets arising from the study of permutations avoiding the pattern 132 and the classical problem of enumerating non-crossing subgraphs in a complete graph drawn on a regular polygon. The methods developed can be extended to enumerate subclasses of permutations avoiding the pattern 1324, the only principal permutation class of length 4 that remains unenumerated.

For any permutation π we can extract its sequence of left-to-right minima (lrm), e.g. if $\pi = 845367912$ we have the lrm-sequence 8431. We can use the lrm-sequence to draw a staircase grid, and place the remaining points of the permutation, 5,6,7,9 and 2, in the boxes of the grid:

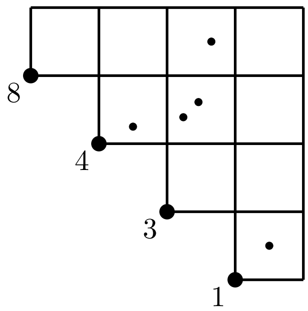

Figure 1: The permutation 845367912 drawn on a staircase grid highlighting its lrm-sequence

If π is a permutation that avoids the (classical permutation) pattern 132, such as the one above, notice that any row contains an increasing sequence of numbers, e.g. the second row contains the increasing sequence 567. Moreover, notice that every rectangular region of boxes is also increasing. The presence of points in a box can force other boxes to be empty. For example the 9 in the third box in row 1 forces the rightmost boxes in rows 2 and 3 to be empty. This exclusion is mutual. We now construct a graph by placing a vertex for every box and an edge between boxes that exclude one another, see the graph on the left in Figure 2.

We can use the graph in the staircase grid to encode the permutation above as the independent set consisting of vertices 13, 22, 23, 44, and the number of points for each vertex $1, 1, 2, 1$.

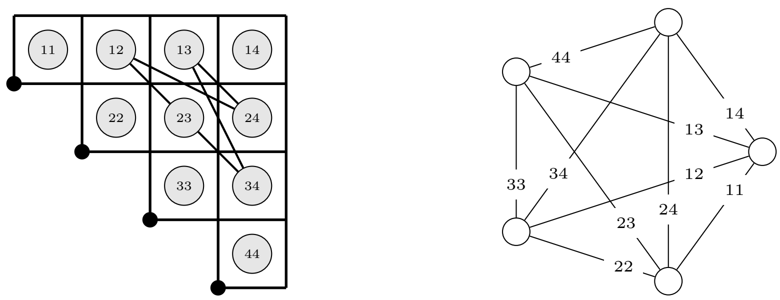

Figure 2: Note that two vertices are connected in the graph on the left if and only if the corresponding edges in the polygon on the right cross

An independent set of k vertices in this graph and a sequence of positive integers of length k determines a unique permutation avoiding 132. Instead of enumerating the independent sets directly we show they are in bijection with the non-crossing subgraphs in a complete graph drawn on a regular polygon; see the graph on the right in Figure 2. Enumerating these non-crossing subgraphs is a classical problem, see e.g. Comtet [1]. For our purposes we use the generating function found by Flajolet and Noy [2]:

$$F(x,y) = 1 + x \cdot F(x,y) + \frac{xy \cdot F(x,y)^2}{1 - y \cdot (F(x,y) - 1)}.$$

Here x marks the lrm's and y marks boxes in the staircase grid (or edges in the polygon). Setting $y = y/(1-y)$, $x = xy$, and collecting by powers of y gives the Narayana triangle enumerating the permutations avoiding 132 by their number of lrm's.

Although it can be shown that no similar bijection exists for 123-avoiding permutations, a similar correspondence can still be established.

We next turn our attention to 1324-avoiding permutations. To get a similar grid structure we must also extract the right-to-left maxima (rlm). E.g. if $\pi = 213679845$ the lrm-sequence is 12 and the rlm-sequence is 985 and we get the grid in Figure 3.

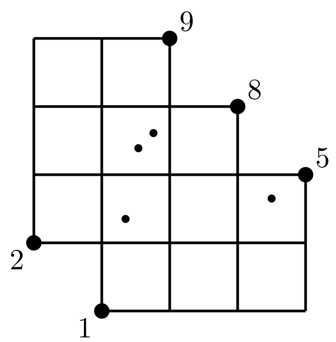

Figure 3: The permutation 213679845 drawn on a staircase grid highlighting its lrm's and rlm's

We say that a permutation has a *non-intersecting boundary of type* (a, b) if it has a left-to-right minima, b right-to-left maxima and these two sequences do not intersect, in the sense that the smallest lrm is to the left of the largest rlm, and the first lrm is smaller than the last rlm. Note that our example above has a non-intersecting boundary of type $(2, 3)$.

Using the same techniques as above we show that there is a generalized polygon whose edges correspond to the vertices in the graph and enumerate the 1324-avoiders with non-intersecting boundaries where *either* a or b is at most 3.

This talk is based on joint work with Christian Bean and Henning Ulfarsson.

References

[1] L. Comtet, Advanced combinatorics, D. Reidel Publishing Co., Dordrecht (1974).

[2] P. Flajolet and M. Noy, Analytic combinatorics of non-crossing configurations, Discrete Math., 204 (1999), 203–229