Counting special inversions in permutations

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December 3, 2010

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Permutations

Definition

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Permutations

Definition

A permutation of rank n is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

- We denote the set of all ($n!$ many) permutations of rank n by $\mathfrak{S}_n$, the permutation group of n.
- A permutation is a finite sequence (a function whose domain is $\{1, \ldots, n\}$). We may write σ_j for $\sigma(j)$.

One-line notation

We will use *one-line notation* for permutations, for example, $\sigma = 32415$ is the permutation in $\mathfrak{S}_5$ that sends

$$\begin{aligned} 1 &\mapsto 3 \\ 2 &\mapsto 2 \\ 3 &\mapsto 4 \\ 4 &\mapsto 1 \\ 5 &\mapsto 5. \end{aligned}$$

Pattern

A classical permutation pattern is a permutation.

Definition

An occurrence of a pattern $\tau : \{1, \dots, k\} \to \{1 \dots, k\}$ in a permutation σ of rank $n \geq k$ is a subsequence $\{\sigma_{m_j}\}_{j=1}^{k}$ of σ that is *order-isomorphic* to τ, meaning that $\tau_i > \tau_j$ iff $\sigma_{m_i} > \sigma_{m_j}$ for each $i, j \in \{1, \dots, k\}$.

Example: the permutation $\sigma = 32415$ has 5 occurrences of the pattern $\tau = 213$:

32415   **32**41**5**   **3**24**15**   3**2**4**15**   32**415**

A motivating observation

By Knuth (1968) *The Art of Computer Programming, Vol. 1*:

Permutations that can be "sorted" by stacks are permutations that avoid the pattern 312.

A permutation σ is *sorted* by a stack if there is a way to place items onto the stack and off again such that the i^{th} item to be placed on is the $\sigma(i)^{\mathrm{th}}$ to be taken off.

Inversions

An inversion in a permutation is a pair of two letters *in the wrong order*. The permutation $\sigma = 32415$ has four inversions.

$$\mathbf{32}415 \quad \mathbf{3}24\mathbf{1}5 \quad 3\mathbf{2}4\mathbf{1}5 \quad 32\mathbf{41}5$$

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We can also describe inversions as occurrences of the classical pattern 21.

Non-inversions

A non-inversion in a permutation are two letters *in the correct order*. The permutation $\sigma = 32415$ has six non-inversions.

32**4**15   **3**241**5**   3**24**15   3**2**41**5**   32**4**1**5**   324**15**

Let $\mathrm{ninv}(\sigma)$ be the number of non-inversions in σ.

We can also describe non-inversions as occurrences of the classical pattern 12.

Generating functions

Let $\mathrm{occ}(\tau, \sigma)$ be the number of occurrences of a pattern τ in the permutation σ. A generating function of rank n for a pattern τ is the polynomial

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{occ}(\tau,\sigma)}.$$

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For each k,

- the coefficient in $A_n(x)$ of x^k is the number of permutations with exactly k occurrences of the pattern τ.

The constant term (the coefficient of x^0) is the number of permutations that avoid the pattern τ.

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Observation

Counting $\operatorname{occ}(\tau, \sigma)$ involves investigating one permutation, while determining each coefficient of $A_n(x)$ involves all permutations in $\mathfrak{S}_n$. It is thus helpful to have an explicit representation of A_n.

Example: The generating function for inversions.

Example

Consider the permutation group $\mathfrak{S}_2 = \{12, 21\}$. Lets build the generating function for number of inversions (occurrences of $\tau = 21$) for this group:

- There is one permutation with zero inversions, namely 12. This contributes $1 \cdot x^0$ to the function.
- There is one permutation with one inversion, namely 21. This contributes $1 \cdot x^1$ to the function.

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Now lets repeat this example for $\mathfrak{S}_3$.

Example

Consider the permutation group $\mathfrak{S}_3$, which consists of

$$123, 132, 213, 231, 312, 321.$$

- There is one permutation with zero inversions.
  This contributes $1 \cdot x^0$ to the function.
- There are two permutations with one inversion.
  This contributes $2 \cdot x^1$ to the function.
- There are two permutations with two inversions.
  This contributes $2 \cdot x^2$ to the function.
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$$A_3(x) = 1 + 2x + 2x^2 + x^3.$$

The generating function for inversions

It is well-known among those who study permutation patterns that

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{inv}(\sigma)} = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

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Another way to look at this is

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Think of j as being n minus the value of σ_n, the number of inversions that last position contributes.

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The same formula holds true for non-inversions as for inversions. We view j as one less than the value of σ_n.

Refining the generating function

Let $\mathrm{NINV}(\sigma)$ be the set of non-inversions in σ. For $(a, b) \in \mathrm{NINV}$, a and b are positions that map to $\sigma(a)$ and $\sigma(b)$.

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Let $\mathrm{NINV}(\sigma)$ be the set of non-inversions in σ. For $(a, b) \in \mathrm{NINV}$, a and b are positions that map to $\sigma(a)$ and $\sigma(b)$.
We can rewrite $\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{ninv}(\sigma)}$ as

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{(a,b) \in \mathrm{NINV}(\sigma)} x,$$

and refine this to consider position and value separations:

$$F_n(x, y, z) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{(a,b) \in \mathrm{NINV}(\sigma)} x y^{b-a} z^{\sigma(b)-\sigma(a)}.$$

Is it possible to give a nice description of this F?

This function can be rewritten as

$$F_n(x, y, z) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{ninv}(\sigma)} y^{\alpha} z^{\beta},$$

where

$$\alpha = \sum_{(a,b) \in \mathrm{NINV}(\sigma)} b - a, \qquad \beta = \sum_{(a,b) \in \mathrm{NINV}(\sigma)} \sigma(b) - \sigma(a).$$

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For example

$$\begin{aligned}
F_2(x,y,z) &= xyz+1,\\
F_3(x,y,z) &= x^3y^4z^4+2x^2y^3z^3+2xyz+1,\\
F_4(x,y,z) &= x^6y^{10}z^{10}+3x^5y^9z^9+x^4y^8z^8+4x^4y^7z^7+2x^3y^6z^6\\
&\quad +2x^3y^5z^5+2x^3y^4z^4+4x^2y^3z^3+x^2y^2z^2+3xyz+1.
\end{aligned}$$

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&\quad + 2x^3y^5z^5 + 2x^3y^4z^4 + 4x^2y^3z^3 + x^2y^2z^2 + 3xyz + 1.
\end{aligned}$$

All terms have y and z raised to the same power!

ninv-sum

Lemma

For any permutation

$$\sum_{(a,b)\in \mathrm{NINV}(\sigma)} b - a = \sum_{(a,b)\in \mathrm{NINV}(\sigma)} \sigma(b) - \sigma(a).$$

If we denote the first sum with ninv-sum(σ), letting σ^{i} be the inverse of σ, the second is

$$\sum_{(\sigma(a),\sigma(b))\in \mathrm{NINV}(\sigma^{\mathrm{i}})} \sigma(b) - \sigma(a) = \text{ninv-sum}(\sigma^{\mathrm{i}}).$$

Then the lemma says

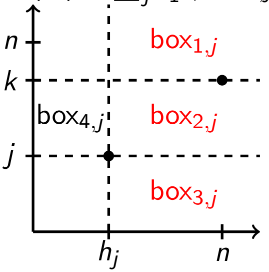
$$\text{ninv-sum}(\sigma) = \text{ninv-sum}(\sigma^{\mathrm{i}}).$$

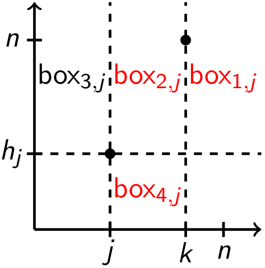
Proving the lemma by induction.

Let σ be an arbitrary permutation and let $\sigma(n) = k$ and $\sigma(h_j) = j$ for $j < k$. Let τ be obtained from σ by removing the last element, $k = \sigma(n)$. Then $\text{ninv-sum}(\tau) = \text{ninv-sum}(\tau^{\mathrm{i}})$. But

$$\text{ninv-sum}(\sigma) = \text{ninv-sum}(\tau) + \sum_{j=1}^{k-1} |\,\text{box}_{1,j}\,| + |\,\text{box}_{2,j}\,| + |\,\text{box}_{3,j}\,| + 1$$

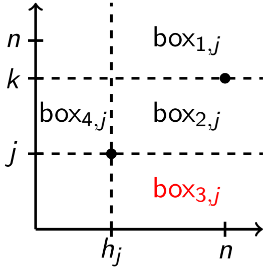
$$\text{ninv-sum}(\sigma^{\mathrm{i}}) = \text{ninv-sum}(\tau^{\mathrm{i}}) + \sum_{j=1}^{k-1} |\,\text{box}_{1,j}\,| + |\,\text{box}_{2,j}\,| + |\,\text{box}_{4,j}\,| + 1.$$

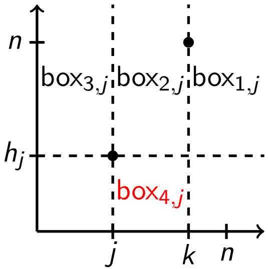




Proving the lemma by induction.

It remains to show that $\sum_{j=1}^{k-1} |\,\mathrm{box}_{3,j}\,| = \sum_{j=1}^{k-1} |\,\mathrm{box}_{4,j}\,|$.





$(a, \sigma(a)) \in \mathrm{box}_{4,\sigma(b)}$ iff
$(a, b) \in \mathrm{INV}$ with $\sigma(a) < k$ iff
$(b, \sigma(b)) \in \mathrm{box}_{3,\sigma(a)}$.

The refined generating function

Because of the lemma it suffices to look at the function

$$G_n(x,y) = F_n(x,y,1) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{ninv}(\pi)} y^{\text{ninv-sum}(\pi)}.$$

Here are some experimental results for $G_n(1,y)$

n	$G_n(1,y)$
2	$y+1$
3	p_4
4	$(y^2+1)p_8$
5	$(y^2-y+1)p_{18}$
6	$(y+1)(y^2-y+1)^2 p_{30}$
7	$(y^2-y+1)p_{54}$
8	$(y^4+1)(y^2-y+1)p_{78}$
9	p_{120}

where p_k is an irreducible polynomial of degree k.

Descents

A descent in a permutation is a pair of adjacent letters *in the wrong order*. The permutation $\sigma = 32415$ has two descents.

32415 32**41**5

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Let $\operatorname{des}(\sigma)$ be the number of descents in σ.

Note that a descent is a special case of an inversion.

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Note that a descent is a special case of an inversion.

Note:

We *cannot* describe a descent as an occurrence of a classical pattern, but we can describe it as an occurrence of a "vincular pattern" which places a restriction on the positions of the subsequence.

Closed formula for the generating function for descents.

For a general n, the generating function for the number of descents is known to be the n^{th} Eulerian polynomial E_n that is often defined recursively by $E_0(x) = 0$, and

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k(x)(x-1)^{n-1-k} = \sum_{k=0}^{n} \frac{n! E_k(x)}{k!(n-k)!}(x-1)^{n-1-k}.$$

Proposition

$$E_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathsf{des}(\sigma)}$$

k-step inversions

A k-step inversion is an an inversion $(a, b) \in \mathrm{INV}$, such that $b - a = k$.

Example

The permutation $\sigma = 32415$ has four inversions

$$\mathbf{32}415 \quad \mathbf{3}24\mathbf{1}5 \quad 3\mathbf{2}4\mathbf{1}5 \quad 32\mathbf{41}5$$

The first is 1-step, second is 3-step, third is 2-step and the last is 1-step.

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Note that a 1-step inversion is a descent.

The generating function for k-step inversions

Let $\mathrm{inv}_k(\sigma)$ be the number of k-step inversions in σ. Then $\mathrm{inv}(\sigma) = \sum_{k=1}^{n-1} \mathrm{inv}_k(\sigma)$. Define

$$H_{n,k}(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{inv}_k(\sigma)}.$$

Let $I(n, k, i)$ represent the coefficient of x^i in $H_{n,k}(x)$, that is, the number of permutations in $\mathfrak{S}_n$ with the number of k-step inversions equalling the number i.

A formula for $H_{n,k}$

Theorem

For $1 \leq k \leq n$ *let* $s = \lfloor n/k \rfloor + 1$ *and* $t = \mathrm{rem}(n/k)$. *If* $k < n/2$

$$H_{n,k}(x) = I(n,k,0)E_s^t(x)E_{s-1}^{k-t}(x),$$

where $E_\ell(x)$ *is the* ℓ^{th} *Eulerian polynomial, the generating function for the number of descents in a permutation.*

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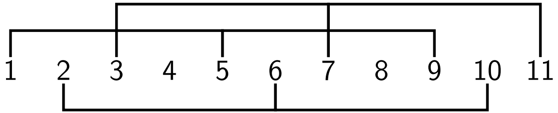
Note that if we let $k = 1$ then the formula in the theorem gives

$$H_{n,1}(x) = I(n,1,0)E_{n+1}^0(x)E_n^1(x) = E_n(x),$$

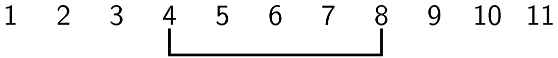
since $I(n,1,0) = 1$. This is to be expected since a 1-step inversion is a descent.

Idea behind the proof

Consider the case $n = 11$, $k = 4$. Consider the following 4 "runs", where 4-step inversions can only occur positions within the same run. Of those 3 are of length 3.



The remaining 1 is of length 2.



This implies that $H_{11,4} = I(11,4,0)E_3^3(x)E_2^1(x)$.

Thank you!