Counting special inversions in permutations

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Table of Contents

We start with some definitions.

Permutations

A **permutation** in $\mathfrak{S}_n$ is a bijection $\pi\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$.
We will use one-line notation for permutations, for example, $\pi = 32415$ is the permutation in $\mathfrak{S}_5$ that sends

$$\begin{aligned} 1 &\mapsto 3 \\ 2 &\mapsto 2 \\ 3 &\mapsto 4 \\ 4 &\mapsto 1 \\ 5 &\mapsto 5. \end{aligned}$$

Descents

A **descent** in a permutation are two adjacent letters *in the wrong order*. The permutation $\pi = 32415$ has two descents.

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We can also describe descents as occurrences of the vincular pattern $\underline{21}$.

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The permutation $\pi = 32415$ has four inversions.

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Note that a descent is a special case of an inversion. The first letter of an inversion is called an **inversion top** and the second letter is called an **inversion bottom**.

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An **inversion** in a permutation are two letters *in the wrong order*. The permutation $\pi = 32415$ has four inversions.

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Note that a descent is a special case of an inversion. The first letter of an inversion is called an **inversion top** and the second letter is called an **inversion bottom**.

We can also describe descents as occurrences of the classical pattern 21.

Non-inversions

A **non-inversion** in a permutation are two letters *in the correct order*. The permutation $\pi = 32415$ has six non-inversions.

$$\mathbf{3}2\mathbf{4}15 \quad \mathbf{3}241\mathbf{5} \quad 3\mathbf{24}15 \quad 3\mathbf{2}41\mathbf{5} \quad 32\mathbf{4}1\mathbf{5} \quad 324\mathbf{15}$$

A non-inversion where the letters are adjacent is called an **ascent**. The first letter of a non-inversion is called an **non-inversion bottom** and the second letter is called a **non-inversion top**.

We can also describe descents as occurrences of the classical pattern 12.

(k, ℓ)-step (non-)inversions

Given a permutation π and an inversion (i, j) in it we say it is a (k, ℓ)-**step inversion** if $j - i = k$ and $\pi(i) - \pi(j) = \ell$. (Exact same definition for non-inversions.)

Example

The permutation $\pi = 32415$ has four inversions

$$\mathbf{32}415 \quad \mathbf{3}24\mathbf{1}5 \quad 3\mathbf{2}4\mathbf{1}5 \quad 32\mathbf{41}5$$

The first is $(1, 1)$-step, second is $(3, 2)$-step, third is $(2, 1)$-step and the last is $(1, 3)$-step.

The generating function for descents.

Generating functions are a convenient way to store a bunch of numbers.

Example

Consider the permutation group $\mathfrak{S}_2 = \{12, 21\}$. Lets build the generating function for number of descents for this group. There is one permutation with zero descents, namely 12. This contributes $1 \cdot x^0$ to the function. There is one permutation with one descent, namely 21. This contributes $1 \cdot x^1$ to the function.

$$A(x) =$$

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Now lets repeat this example for $\mathfrak{S}_3$.

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Example

Consider the permutation group $\mathfrak{S}_3$, which consists of

$$123, 132, 213, 231, 312, 321.$$

Lets build the generating function for number of descents for this group. There is one permutation with zero descents. This contributes $1 \cdot x^0$ to the function. There are four permutations with one descent. This contributes $4 \cdot x^1$ to the function. There is one permutations with two descents. This contributes $1 \cdot x^2$ to the function.

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$$A(x) = 1 + 4x + x^2.$$

The generating function for descents.

For a general n we get the n-th Eulerian polynomial as the generating function for the number of descents.

$$E_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)}.$$

We now want to focus specifically on $(2, 2)$-step non-inversions.

Counting permutations with $(2,2)$-step inversions

Empirical testing shows that the number of permutations that contain at least one $(2,2)$-step non-inversion (starting from rank 1) is

$$0, 0, 1, 6, 45, 310, 2311, 19414, \ldots .$$

We aim to provide a formula for these numbers.

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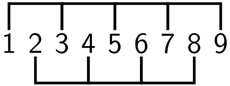
Example

Consider for example the permutation 153769482 which contains three $(2,2)$-step non-inversions ($\pi(1,3)=(1,3)$, $\pi(2,4)=(5,7)$ and $\pi(4,6)=(7,9)$).

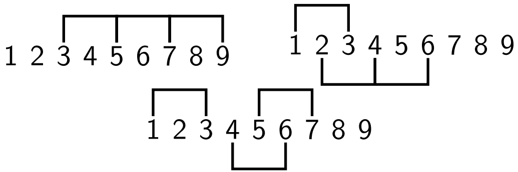
The odd class and the even class

The *odd class* consists of pairs of the form $(x, x+2)$ where x is an odd number and the *even class* consists of pairs of the same form with x an even number.



Selecting the pairs

We begin with an example of where we would like to select $j = 3$ pairs of positions of the form $(x, x+2)$ inside a permutation. There are three distinct ways:



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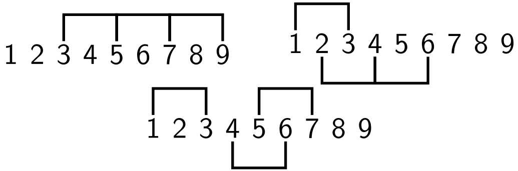
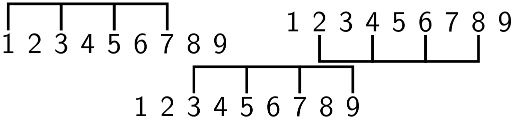


Each way corresponds to a partition of 3: 3, $2+1$, $1+1+1$. Note also that the number of elements involved in the pairs is $4, 5, 6$, respectively.

Selecting the pairs

Given a permutation π of length n, for each partition λ of 3, let $P(n, \lambda)$ represent the number of ways of realizing the partition inside the permutation. For example if $\lambda = 3$ then $P(9, \lambda) = 3$ as can be seen below.



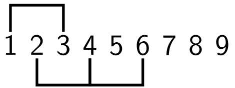
Selecting the pairs

For a general j the distinct ways of finding j pairs inside a permutation are in bijection with the partitions of j. For a fixed partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of j we then have to decide which of the components of λ go into the odd class and which go into the even class. Write $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_r}$ for the unique numbers appearing in λ and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be their multiplicities. Let $m_\lambda = \prod_j \alpha_j!$, and $\ell_\lambda = \ell$, the number of components of λ.
Let us define

$$Q(n,j) = \sum_{\lambda \dashv j} P(n,\lambda)^2 \cdot m_\lambda \cdot (n - j - \ell_\lambda)!.$$

Overcounting

But $Q(n,1)$ does not count the number of permutations of rank n that contain at least one $(2,2)$-step non-inversion because most permutations are counted more than once. Consider for example the permutation 153769482 which contains three $(2,2)$-step non-inversions ($\pi(1,3)=(1,3)$, $\pi(2,4)=(5,7)$ and $\pi(4,6)=(7,9)$). It will be counted three times in $Q(9,1)$. It will also be counted three times in $Q(9,2)$, twice when $\lambda = 1+1$ and once when $\lambda = 2$. Finally it will also be counted once in $Q(9,3)$ when $\lambda = 2+1$.



The final formula

Now notice that this particular permutation contributes 1 to the sum

$$Q(9,1) - Q(9,2) + Q(9,3) - \dots .$$

This cancellation works in general:

Theorem

The number of permutations in $\mathfrak{S}_n$ that contain a $(2,2)$-step non-inversion is given by

$$\sum_{j=1}^{n-2} (-1)^{j+1} Q(n,j)$$

The next few values

With this formula we can calculate the next few values of the function

$$181381, 1865310, 20973099, 256179022, 3379395901, 47895552166, 725972592631, 11720476777494, 200813523247197, 3639573082928638.$$

We know only pay attention to the displacement in location in the (non)-inversions.

k-step inversions

A k-**step inversion** is a (k, ℓ)-step inversion for some ℓ.

Example

The permutation $\pi = 32415$ has four inversions

$$\mathbf{32}415 \quad \mathbf{3}24\mathbf{1}5 \quad 3\mathbf{2}4\mathbf{1}5 \quad 32\mathbf{41}5$$

The first is 1-step, second is 3-step, third is 2-step and the last is 1-step.

A generating function

It is a well-known fact that

$$\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{ninv}(\pi)} = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}) = [n]_x!.$$

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Now this is also equal to

$$\sum_{\pi\in\mathfrak{S}_n} \prod_{(a,b)\in\mathrm{NINV}(\pi)} x.$$

We can refine this into

$$F_n(x,y,z) = \sum_{\pi\in\mathfrak{S}_n} \prod_{(a,b)\in\mathrm{NINV}(\pi)} xy^{b-a}z^{\pi(b)-\pi(a)}.$$

Is it possible to give a nice description of this F?

A generating function

This function can be rewritten as

$$F_n(x, y, z) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{ninv}(\pi)} y^{\alpha} z^{\beta},$$

where

$$\alpha = \sum_{(a,b) \in \mathrm{NINV}(\pi)} b - a, \qquad \beta = \sum_{(a,b) \in \mathrm{NINV}(\pi)} \pi(b) - \pi(a).$$

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For example

$$\begin{aligned} F_2(x, y, z) &= xyz + 1, \\ F_3(x, y, z) &= x^3y^4z^4 + 2x^2y^3z^3 + 2xyz + 1, \\ F_4(x, y, z) &= x^6y^{10}z^{10} + 3x^5y^9z^9 + x^4y^8z^8 + 4x^4y^7z^7 + 2x^3y^6z^6 \\ &\quad + 2x^3y^5z^5 + 2x^3y^4z^4 + 4x^2y^3z^3 + x^2y^2z^2 + 3xyz + 1. \end{aligned}$$

All terms have y and z raised to the same power!

inv-sum and ninv-sum

Lemma

For any permutation

$$\sum_{(a,b)\in \mathrm{NINV}(\pi)} b - a = \sum_{(a,b)\in \mathrm{NINV}(\pi)} \pi(b) - \pi(a).$$

If we denote the first sum with ninv-sum(π), then the lemma says

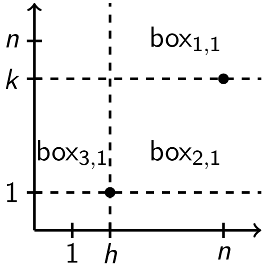
$$\text{ninv-sum}(\pi) = \text{ninv-sum}(\pi^{\mathrm{i}}).$$

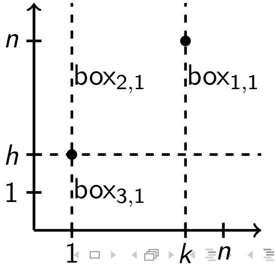
Proving the lemma by induction.

Let π be an arbitrary permutation and let $\pi(n) = k$ and $\pi(h) = 1$. Let τ be the permutation obtained from π by removing the last element, $k = \pi(n)$. Then ninv-sum(τ) = ninv-sum(τ^{i}). But

$$\text{ninv-sum}(\pi) = \text{ninv-sum}(\tau) + \sum_{i=1}^{k-1} |\,\text{box}_{1,i}\,| + |\,\text{box}_{2,i}\,|$$

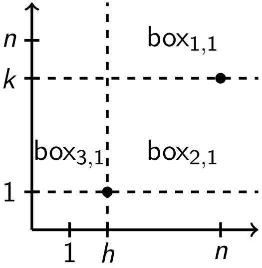
$$\text{ninv-sum}(\pi^{\mathrm{i}}) = \text{ninv-sum}(\tau^{\mathrm{i}}) + \sum_{i=1}^{k-1} |\,\text{box}_{1,i}\,| + |\,\text{box}_{2,i}\,| + |\,\text{box}_{3,i}\,| - i.$$

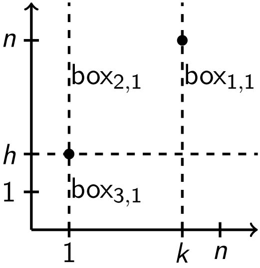




Proving the lemma by induction.

Finally it is easy to see that $\sum_{i=1}^{k-1} |\,\mathrm{box}_{3,i}\,| - i = 0$ so we are done.





The refined generating function

Because of the lemma it suffices to look at the function

$$G_n(x, y) = F_n(x, y, 1) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{ninv}(\pi)} y^{\mathrm{ninv\text{-}sum}(\pi)}.$$

Here are some experimental results for $G_n(1, y)$

n	Small factors
1	1
2	$y+1$
3	1
4	(y^2+1)
5	(y^2-y+1)
6	$(y+1)(y^2-y+1)^2$
7	(y^2-y+1)
8	$(y^4+1)(y^2-y+1)$
9	1

The cosine of the permutation

Definition

For a permutation π of rank n the number

$$\mathbf{1} \cdot \pi = \sum_{i=1}^{n} i\pi(i)$$

is called the **cosine** of the permutation.

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Note that if we treat the permutations as vectors then

$$\mathbf{1} \cdot \pi = |\mathbf{1}| \cdot |\pi| \cos(\theta) = \frac{n(n+1)(2n+1)}{6} \cos(\theta),$$

so $\mathbf{1} \cdot \pi$ only depends on the cosine of the angle between the identity and the permutation.

The cosine and the non-inversion sum

If we calculate the cosine and the non-inversions sum for a few random permutations we get

π	$\mathbf{1} \cdot \pi$	ninv-sum(π)
41352	45	10
21435	53	18
24513	44	9
25134	47	13

we might begin to suspect that the cosine and the non-inversion sum always differ by the same number.

The cosine and the non-inversion sum

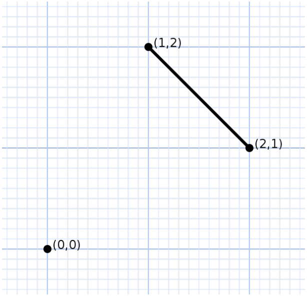
Theorem

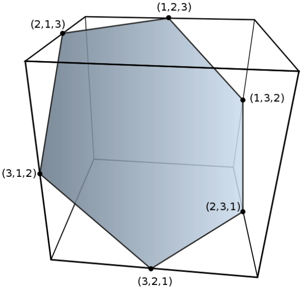
For any permutation π,

$$\mathbf{1} \cdot \pi = \mathbf{1} \cdot \mathbf{1}^{\mathrm{r}} + \text{ninv-sum}(\pi).$$

The permutohedron

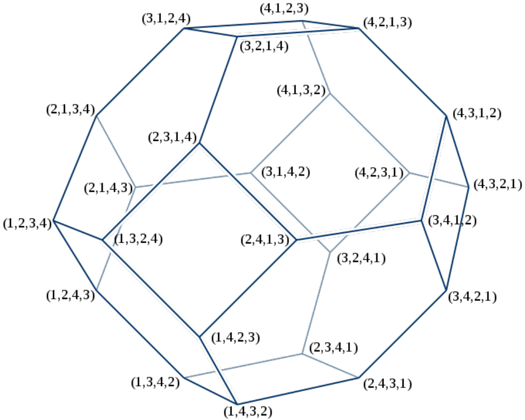
Start with the vertex $(1, 2)$ and form all permutations of its coordinates. This will give the permutohedron of order 2. If you do this for $(1, 2, 3)$ you get the permutohedron of order 3.





The permutohedron

Here is the permutohedron of order 4.



The permutohedron

The previous theorem says that

$$\mathbf{1}\cdot\mathbf{1}^{\mathrm{r}} + \text{ninv-sum}(\pi) = \mathbf{1}\cdot\pi = \frac{n(n+1)(2n+1)}{6}\cos(\theta).$$

So the cosine of the angle between the vertex represented by π and the identity vertex $(1, 2, 3, \ldots, n)$ only depends on the sum over the non-inversions of π.

The relationship between the cosine and the non-inversion sum

The motivation for considering the cosine comes from the following question. Given an integer n is it always possible to find a permutation such that $\mathbf{1} \cdot \pi = n$? (See A135298 on the OIES). It seems to be true for $n \geq 35$.

The relationship between the cosine and the non-inversion sum

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We haven't been able to show this. We are trying to build permutations with direct sums and skew sums. Then:

$$\text{ninv-sum}(\pi \ominus \rho) = \text{ninv-sum}(\pi) + \text{ninv-sum}(\rho)$$

If the rank of π is n and the rank of ρ is m then

$$\text{ninv-sum}(\pi \oplus \rho) = \text{ninv-sum}(\pi) + \text{ninv-sum}(\rho) + \frac{mn}{2}(m+n).$$

The generating function for k-step inversions

Let $\mathrm{inv}_k(\pi)$ be the number of k-step inversions in π. Then $\mathrm{inv}(\pi) = \sum_{k=1}^{n-1} \mathrm{inv}_k(\pi)$. Define

$$H_{n,k}(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{inv}_k(\pi)}$$

$$I(n,k,i) = [x^i] H_{n,k}(x)$$

represent the number of permutations in $\mathfrak{S}_n$ with a number of k-step inversions equalling the number i.

A formula for $H_{n,k}$

Theorem

For $1 \leq k \leq n$ let $s = \lfloor n/k \rfloor + 1$ and $t = \mathrm{rem}(n/k)$. If $k < n/2$

$$H_{n,k}(x) = I(n,k,0)E_s^t(x)E_{s-1}^{k-t}(x),$$

where $E_\ell(x)$ is the ℓ^{th} Eulerian polynomial, the generating function for the number of descents in a permutation.

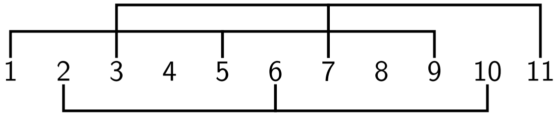
Note that if we let $k = 1$ then the formula in the theorem gives

$$H_{n,1}(x) = I(n,1,0)E_{n+1}^0(x)E_n^1(x) = E_n(x),$$

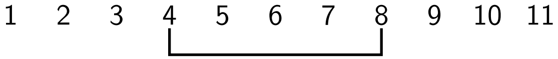
since $I(n,1,0) = 1$. This is to be expected since a 1-step inversion is a descent.

Idea behind the proof

Consider the case $n = 11$, $k = 4$. There are 4 runs in total. Of those 3 are of length 3.



The remaining 1 is of length 2.



This implies that $H_{11,4} = I(11, 4, 0)E_3^3(x)E_2^1(x)$.

Another generating function

Let $\mathrm{inv}_{\leq k}(\pi)$ be the number of k'-step inversions in π for all $k' \leq k$. Then

$$\mathrm{inv}(\pi) = \mathrm{inv}_{\leq n}(\pi), \qquad \mathrm{des}(\pi) = \mathrm{inv}_{\leq 1}(\pi),$$

for any permutation of rank n. Define

$$J_{n,\leq k}(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{inv}_{\leq k}(\pi)}.$$

So in particular we have

$$J_{n,\leq n}(x) = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots x^{n-1})$$

and

$$J_{n,\leq 1}(x) = E_n(x),$$

the n^{th} Eulerian polynomial.

A conjecture

We conjecture that the pattern

$$J_{3,\leq 1}(x) = x^2 + 4x + 1$$
$$J_{4,\leq 2}(x) = (x+1)(x^4 + 2x^3 + 6x^2 + 2x + 1)$$
$$J_{5,\leq 3}(x) = (x+1)(x^2 + x + 1)(x^6 + 2x^5 + 3x^4 + 8x^3 + 3x^2 + 2x + 1)$$

continues (it does, at least up to $J_{7,\leq 5}$).

We now consider the relationship between our work and previous work of Dukes-Reifergerste and Kitaev-Remmel.

Certified non-inversions

We recall some definitions from Dukes, Reifergerste, *The area above the Dyck path of a permutation*: A **certified non-inversion** in π is a non-inversion (a, b) in π such that there is at least one position $a < c < b$ such that $\pi(c) > \pi(a), \pi(b)$.

Certified non-inversions

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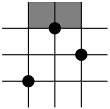

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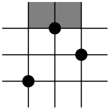

They then define $\mathrm{lbsum}(\pi)$ as the number of inversions in π together with the number of certified non-inversions.

Certified k-step non-inversions

We now define $\mathrm{lbsum}_k(\pi)$ as the number of k-step inversions in π together with the number of certified k-step non-inversions.
Dukes and Reifergerste defined

$$K_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{lbsum}(\pi)}$$

and proved a recurrence formula for it. We consider

$$K_{n,k}(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{lbsum}_k(\pi)}$$

and haven't been able to find a recurrence formula for it.

Special cases of $K_{n,k}(x)$

Theorem

The number of permutations of rank n with lbsum_{n-2} *equal to* 2 *is*

$$(n-2)!(n^2-3n+1).$$

Thus

$$\frac{K_{n,n-2}(x)}{(n-2)!} = (n^2-3n+1)x^2 + 2(n-1)x + 1.$$

Fixing the parity

In *Classifying descents according to parity* and *Classifying descents according to equivalence mod k*, Kitaev and Remmel studied descents starting with an even number and more generally descents starting with a number that is zero modulo some integer k. This leads to the following definition:

Fixing the parity

In *Classifying descents according to parity* and *Classifying descents according to equivalence mod k*, Kitaev and Remmel studied descents starting with an even number and more generally descents starting with a number that is zero modulo some integer k. This leads to the following definition:

Let $\mathrm{modinv}_{d,k}(\pi)$ be the number of k-step inversions with inversion top that is zero modulo d. Let

$$L_{n,d,k}(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{modinv}_{d,k}(\pi)}$$

Fixing the parity

Theorem

The leading coefficient for $k = n-1$ and $d = 2$ equals

$$\left\lfloor \frac{n}{2} \right\rfloor^2 (n-2)!.$$

Thus

$$\frac{L_{n,2,n-1}(x)}{(n-2)!} = \left\lfloor \frac{n}{2} \right\rfloor^2 x + n(n-1) - \left\lfloor \frac{n}{2} \right\rfloor^2.$$

Fixing the parity

Proof.

The formula for the leading coefficient is proved as follows: In order to have one $(n-1)$-step inversion with an even descent top a permutation must start with an even number and end in some smaller number. Thus we get the formula

$$(n-2)! \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (2j-1).$$

Simplification yields the claimed formula. $\square$

Thank you for your time!