Wilf Classification of Mesh Patterns of Short Length

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Permutations

Definition

Let A be a finite, non-empty set. A one-to-one correspondence from A to itself is a permutation. Let $A = \{1, 2, \ldots, n\}$ and denote a permutation as a word, $\pi = \pi_1\pi_2\ldots\pi_n$. Let S_n be the set of all permutations of length n.

Example

The word $\pi = 1324$ is a permutation of the set $\{1, 2, 3, 4\}$ with $\pi_1 = 1$, $\pi_2 = 3$, $\pi_3 = 2$ and $\pi_4 = 4$.

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A classical pattern is a permutation in S_k.

Example

The pattern 231 can be drawn as follows, where the horizontal lines represent the values and the vertical ones denote the locations in the pattern.

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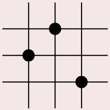

Occurrence/avoidance of patterns

Definition

We say that a pattern occurs in a permutation if there is a subsequence whose letters are in the same relative order of size as the letters of the pattern. If a pattern does not occur in a permutation, the permutation avoids the pattern.

Example

The permutation 362451 contains the pattern 231 =

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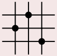
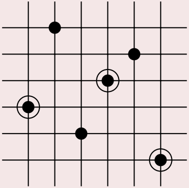
The permutation 362451 contains the pattern 231 = 



Vincular patterns

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For a vincular pattern to occur in a permutation, the pattern requires letters to be adjacent in the permutation.

Example

The pattern

requires the letters corresponding to 1 and 3 in a permutation to be adjacent.

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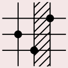

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Bivincular patterns

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A bivincular pattern is a pattern that puts constraints on positions and values in a permutation.

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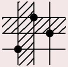

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Mesh patterns

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A mesh pattern is a pair (τ, R), where τ is a permutation in S_k and R is a subset of $[\![0,k]\!] \times [\![0,k]\!]$.

Example

The permutation 165342 contains the pattern

$(132, \{(1,1),(1,2),(2,3),(3,3)\}) =$

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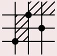
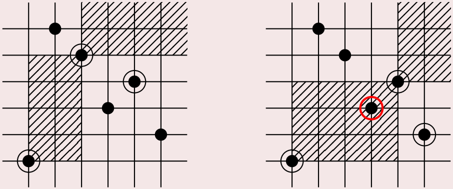
$(132, \{(1,1), (1,2), (2,3), (3,3)\}) =$ 



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Definition

Two patterns p and q are Wilf-equivalent if equally many permutations of length n avoid p and q, for all n.

Wilf-equivalence is one of the big questions in the study of patterns.

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Existing research on Wilf-equivalence

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- We have been studying patterns of length 2
- The number of mesh patterns of length 2 is 1024
- We used Sage to help us sort the patterns by Wilf-equivalence
  - Reverse
  - Inverse
  - Complement
  - Toric
  - Up-shift
  - Shading Lemma
- Using these operations brings us down to 65 classes.
- Robert Parviainen had already Wilf-classified bivincular patterns of length 2 and 3, and thus there are 58 equivalence classes left unproved.

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Lemma (Shading Lemma)

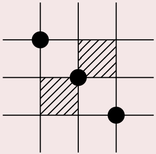
Let (τ, R) be a mesh pattern of length n such that $\tau(i) = j$ and $[i, j] \notin R$. If all of the following conditions are satisfied:

- *The box $[i-1, j-1]$ is not in R;*
- *At most one of the boxes $[i, j-1]$, $[i-1, j]$ is in R;*
- *If the box $[\ell, j-1]$ is in R ($\ell \neq i-1, i$) then the box $[\ell, j]$ is also in R;*
- *If the box $[i-1, \ell]$ is in R ($\ell \neq j-1, j$) then the box $[i, \ell]$ is also in R;*

then the patterns (τ, R) and $(\tau, R \cup \{[i, j]\})$ are equivalent. Analogous conditions determine if the other neighboring boxes can added to R.

Example

The following equivalence is found by using Shading Lemma



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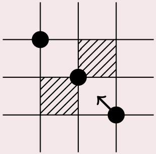

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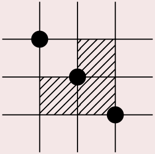

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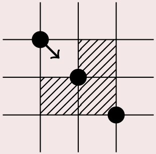

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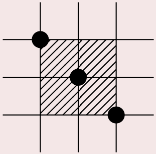

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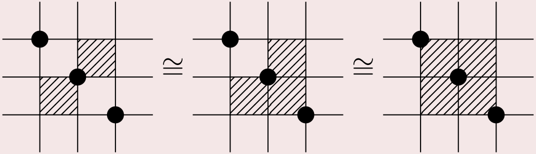
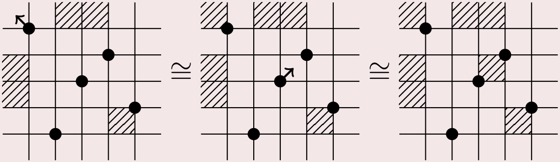
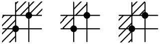
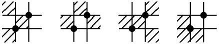
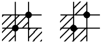


Example

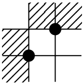
This equivalence is also found by using Shading Lemma



Representative	Formula	# of patterns
	1	28
	$(n-1)!$	40
	$a_n = n \cdot a_{n-1} - a_{n-2}$	32
	$a_n = (n-1)a_{n-1} + (n-2)a_{n-2}$	32
	$[x^n]\left(1 - \frac{1}{\sum_n n!x^n}\right)$	4
	$\sum_{i=1}^{n-1} \frac{(n-1)!}{i}$	84
	$[\frac{x^n}{n}] \log\left(1 + \sum_{i=1}^{n}(i-1)! \cdot x^i\right)$	60

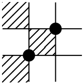
Representative	Formula	# of patterns
	$n! - \sum_{i=1}^{n-1} \sum_{\ell=1}^{i} (i-\ell)!(n-i-\ell)!\ell!$	4
	$n! - \sum_{k=0}^{n-2} \sum_{j=0}^{k} j!(k-j)!(n-2-k)!$	4
	$n! - (n-1)! + [x^n]\frac{F(x)}{1+xF(x)}$	8
	$n! - \sum_{i=0}^{n-2} i!(n-1-i)!$	16
	$n! - \sum_{k=1}^{n-1}(k-1)!(n-k-1)!$	24

The number of permutations avoiding the following pattern



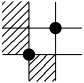
is 1 for all n.

The number of permutations that avoid the pattern



is $(n-1)!$ for all n.

The number of permutations avoiding the pattern

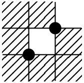


is given by the recursive formula

$$a_n = (n-1)a_{n-1} + (n-2)a_{n-2}$$

where $a_0 = a_1 = 1$.

The number of permutations that avoid the pattern

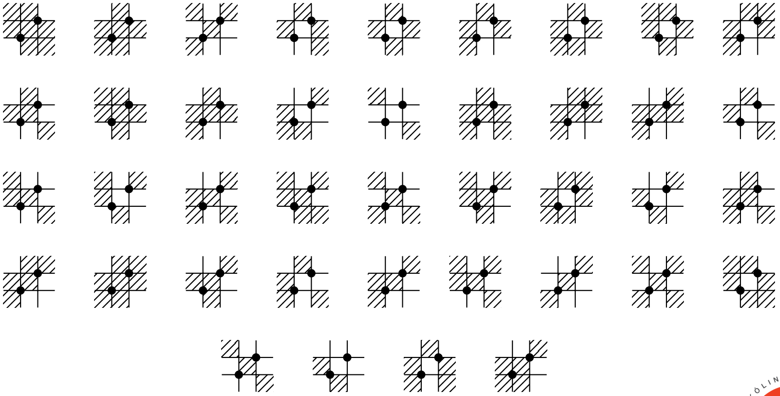


is given by the formula

$$n! - (n-1)! + [x^n]\frac{F(x)}{1 + xF(x)}.$$

where $F(x) = \sum_{n \geq 1} (n-1)! x^{n-1}$.

The pattern classes that we have not found a formula for.



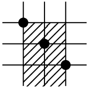
Simsun permutations

Definition

A permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ is called *simsun* if the restriction of π to $\{1, 2, \ldots, k\}$, for all $3 \geq k \geq n$, has no double descents.

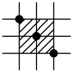
Simsun permutations were first introduced by Sundaram in 1994

A permutation is simsun if and only if it avoids the pattern

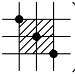
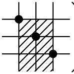


Brändén and Claesson, and Úlfarsson simultaneously and independently showed this connection

We have been studying the following pattern,



which is the only non-trivial interval pattern of length 3. It is not difficult to see that

S_n () $\subseteq$ S_n ()

This gives us a chance to use known results on simsun permutations, such as bijections to binary trees and 1-2-trees.

Outcome

- Intro article about pattern avoidance in Verpill
- New translations added to the Icelandic mathematical dictionary
- Code for Sage
- New sequences in The On-Line Encyclopedia of Integer Sequences (OEIS)
- Research paper submitted to arXiv
- Accepted to Permutation Patterns 2011, a conference in California June 20-24
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Any questions?